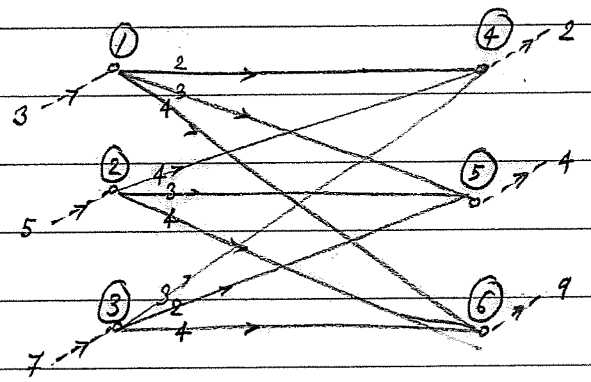
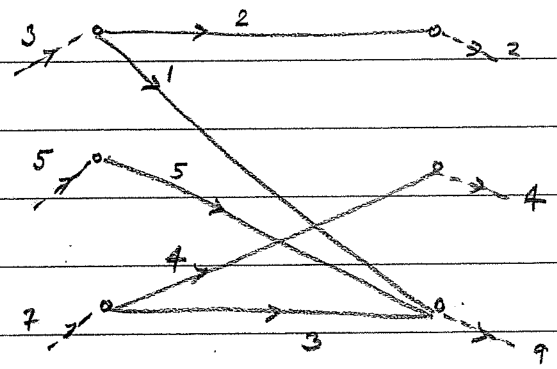
Exercise 7.2

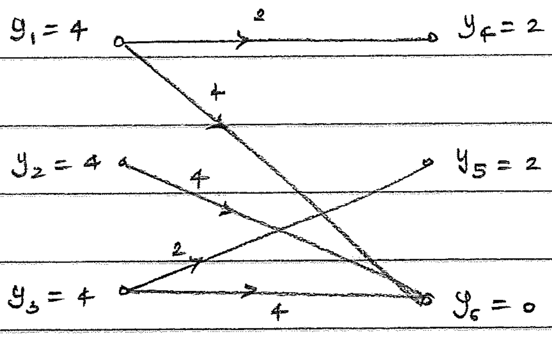
That the problem is a network flow problem follows from the fact that every column in $\tilde{A}$ has one +1, one −1, and the rest are zeros. Every row in $\tilde{A}$ then corresponds to a node in the network, and every column in $\tilde{A}$ corresponds to an edge in the network (namely the edge from the node corresponding to the +1 entry to to the node corresponding to the −1 entry).
Thus the network has 6 nodes and 9 edges, and is shown below:



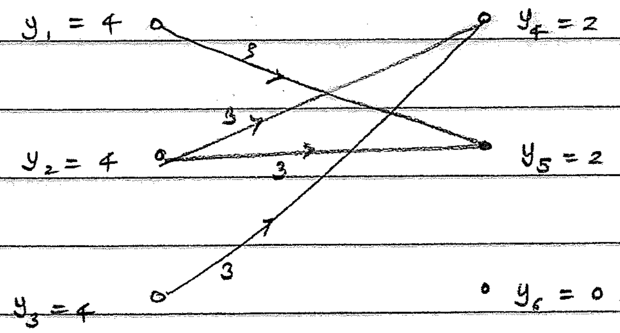
The given solution is a basic feasible solution corresponding to the spanning tree shown below:



The vector y can be computed using $y_i - y_j = c_{ij}$ for all edges (i,j) in the above spanning tree, and with $y_6 = 0$.
So we have:



The reduced costs for the nonbasic variables can be computed using $r_{ij} = c_{ij} - (y_i - y_j)$ for all edges (i,j) which don't belong to the above spanning tree. This gives:



$r_{15} = 3 - (4-2) = 1 \geq 0$,

$r_{24} = 3 - (4-2) = 1 \geq 0$,

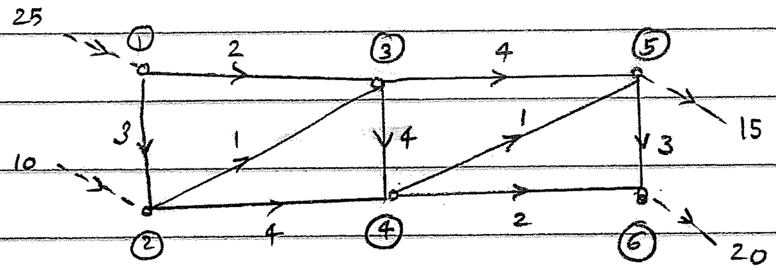
$r_{25} = 3 - (4-2) = 1 \geq 0$,

$r_{34} = 3 - (4-2) = 1 \geq 0$.

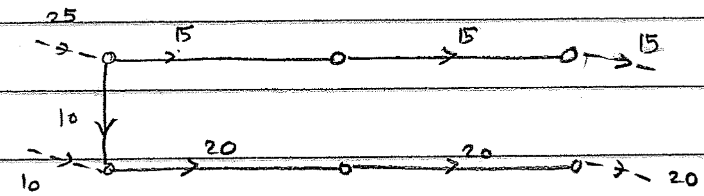
Since all $r_{ij} \geq 0$, the given basic feasible solution is optimal.

Exercise 7.3

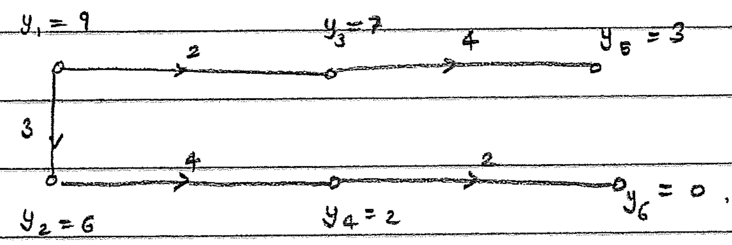
The given network is shown below:



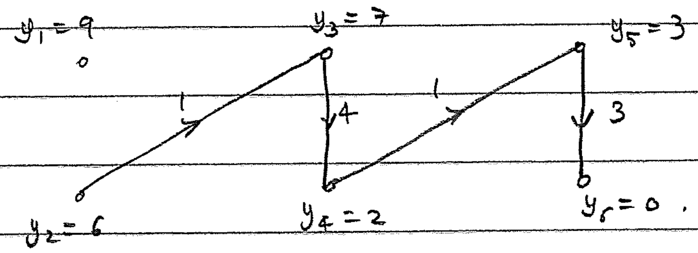
The given basic feasible solution corresponds to the flow in the following spanning tree:



The vector y can be computed using $y_i - y_j = c_{ij}$ for all edges (i,j) in the above spanning tree, and with $y_6 = 0$. So we have:



The reduced costs for the nonbasic variables can be computed using $r_{ij} = c_{ij} - (y_i - y_j)$ for all edges (i,j) which don't belong to the above spanning tree. This gives



$r_{23} = 2 \geq 0$

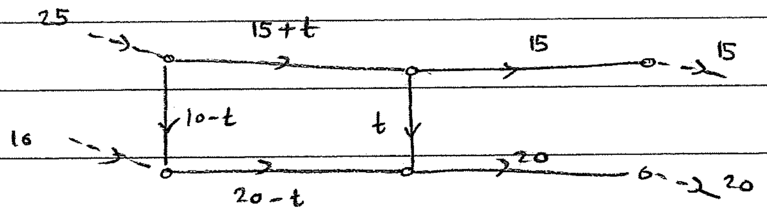
$r_{34} = -1 < 0$

$r_{45} = 2 \geq 0$

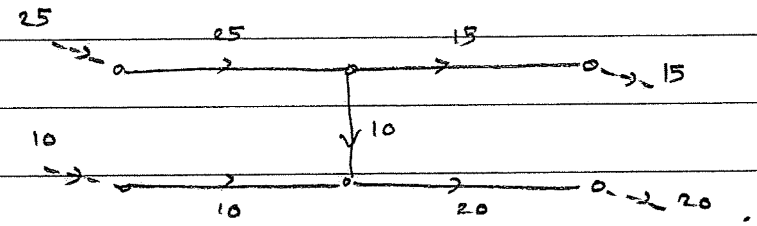
$r_{56} = 0$.

Since $r_{34} < 0$, this solution is not optimal.

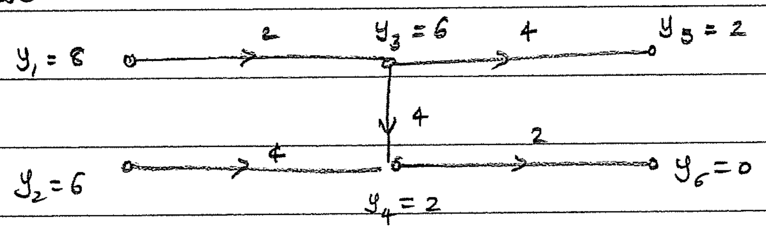
So we let x_{34} become a new basic variable. To this end, we set $x_{34} = t$, and keep the other nonbasic variables at 0. Then the updated values of the basic variables can be found out:



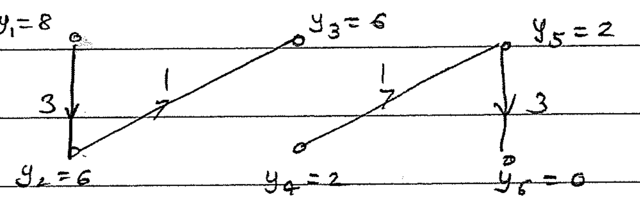
So t can increase up to 10. Hence x_{12} leaves the set of basic variables. The new basic solution is the flow in the following new spanning tree:



The vector y is found using $y_i - y_j = c_{ij}$ for all edges (i,j) in the above spanning tree, and with $y_6 = 0$. So we have:



The reduced costs for the non basic variables can be computed using $r_{ij} = c_{ij} - (y_i - y_j)$ for all edges (i,j) which don't belong to the above spanning tree. This gives:



$r_{12} = 1 \geq 0$

$r_{23} = 1 \geq 0$

$r_{45} = 1 \geq 0$

$r_{56} = 1 \geq 0$.

Since all $r_{ij} \geq 0$, the current basic feasible solution is optimal.

The optimal solution is

$x_{13} = 25$

$x_{12} = 0$

$x_{23} = 0$

$x_{24} = 10$

$x_{34} = 10$

$x_{35} = 15$

$x_{45} = 0$

$x_{46} = 20$

$x_{56} = 0$.

The optimal cost is

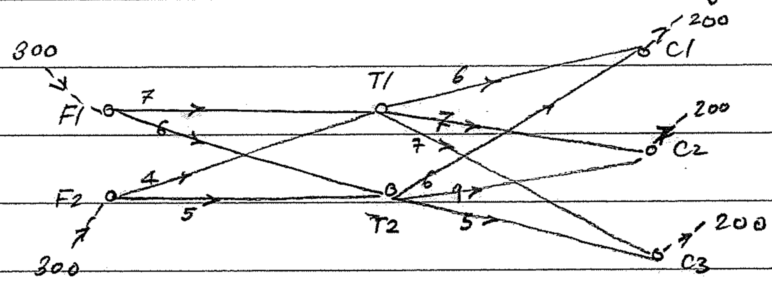
$2 \cdot 25 + 4 \cdot 10 + 4 \cdot 10 + 2 \cdot 20 + 4 \cdot 15$

$= 50 + 40 + 40 + 40 + 60$

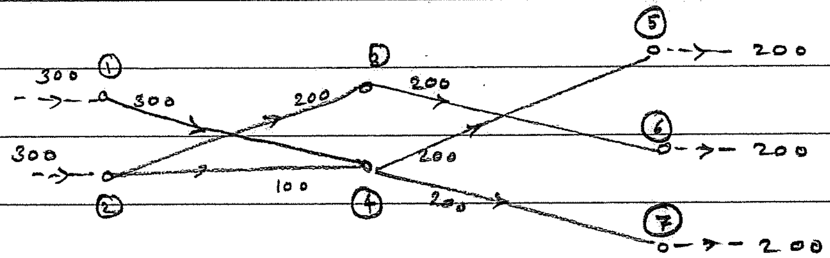
$= 230$.

Exercise 7.4

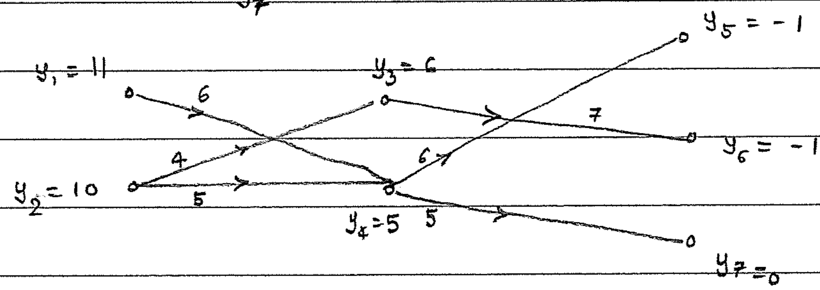
The problem can be formulated as a network flow problem with 7 nodes and 10 edges.



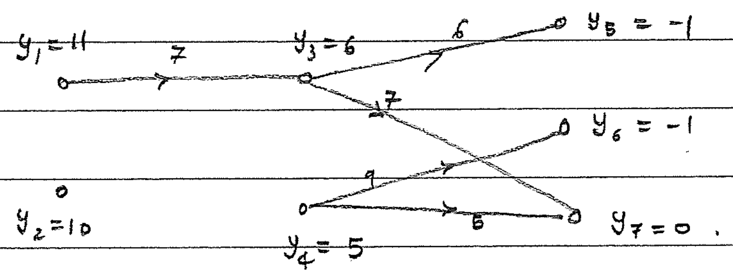
The proposed plan corresponds to a spanning tree (and hence it is a basic feasible solution):



We compute the vector y using the relation $y_i - y_j = c_{ij}$ for the edges (i,j) in the spanning tree above and with $y_7 = 0$.



Next we compute the reduced costs for the nonbasic variables using $r_{ij} = c_{ij} - (y_i - y_j)$ for all (i,j) which are not tree edges in the spanning tree above. This gives:



$r_{13} = 2 \geq 0$

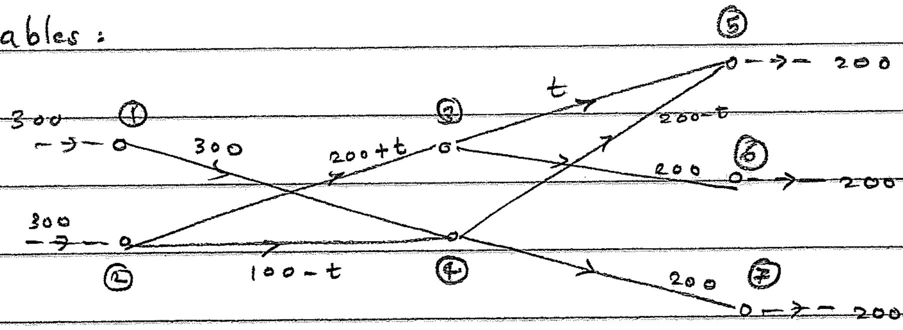
$\boxed{r_{35} = -1 < 0}$

$r_{46} = 3 \geq 0$

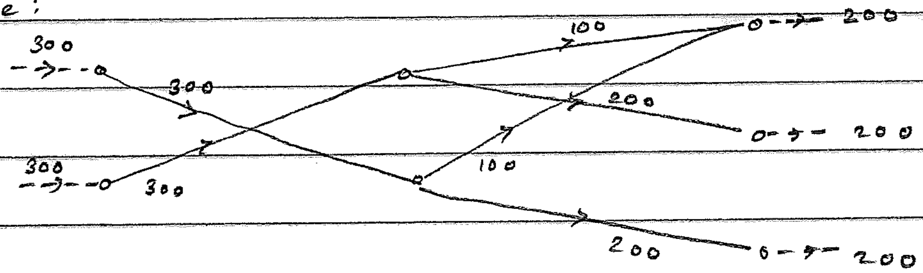
$r_{47} = 0 \geq 0$.

So the proposed plan is not optimal (since $r_{35} = -1 < 0$).

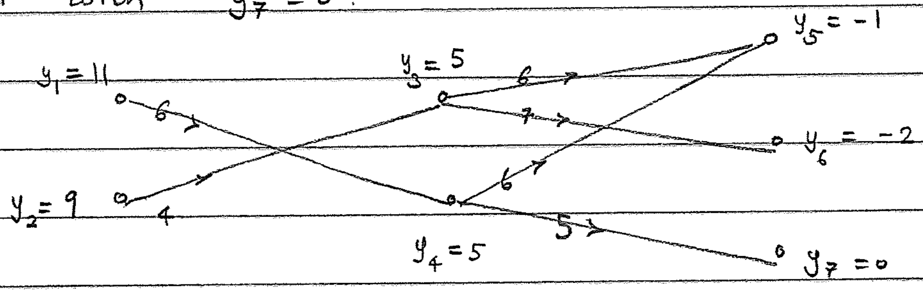
We let x_{35} become a new basic variable. To this end, we set $x_{35} = t$. We keep the other nonbasic variable values at 0, and find the updated values of the basic variables:



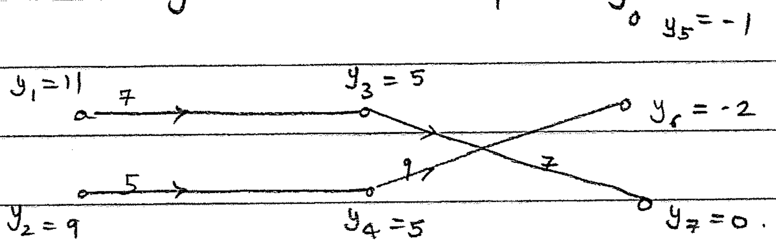
So x_{35} can increase upto 100, and x_{24} leaves the set of basic variables. The new basic feasible solution is the flow in the following new spanning tree:



The vector y can be computed using $y_i - y_j = c_{ij}$ for the edges (i,j) in the spanning tree above and with $y_7 = 0$.



Next we compute the reduced costs for the non basic variables using $r_{ij} = c_{ij} - (y_i - y_j)$ for all (i,j) which are not tree edges in the spanning tree above. This gives:



$r_{13} = 1 \geq 0$,

$r_{24} = 1 \geq 0$,

$r_{37} = 2 \geq 0$,

$r_{45} = 2 \geq 0$.

Since all $r_{ij} \geq 0$ this solution is optimal.

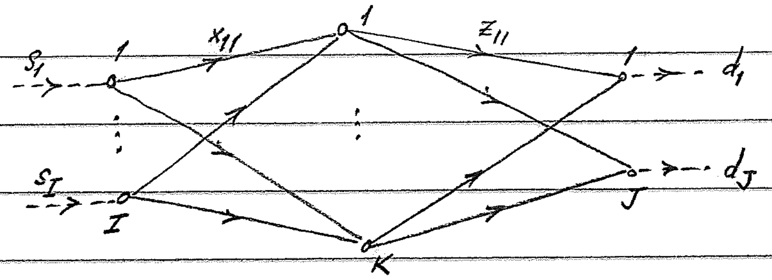
Hence the optimal transport plan is given as follows:

	T1	T2
F1	0	300
F2	300	0

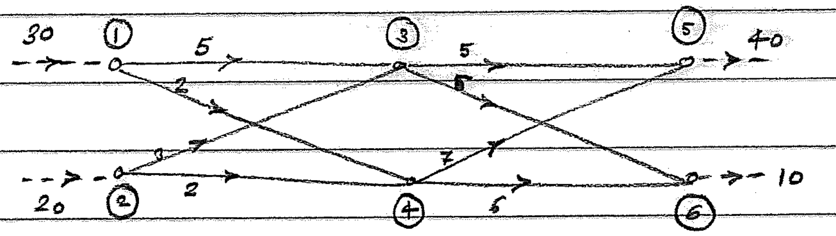
	C1	C2	C3
T1	100	200	0
T2	100	0	200

Exercise 7.5

General network form:



(a) Specific network in the problem:



Our network has 6 nodes, and we let nodes 1, 2 be source nodes, nodes 3 and 4 the intermediate nodes, and finally nodes 5, 6 the sink nodes. The set of edges is:

$$\{(1,3),(1,4),(2,3),(2,4),(3,5),(3,6),(4,5),(4,6)\}.$$

The network flow problem can be written as:

$$(\text{NFP}): \begin{cases} \text{minimize} & c^T v \\ \text{subject to} & Av = b, \\ & v \geq 0, \end{cases}$$

where

$$v = [v_{13} \quad v_{14} \quad v_{23} \quad v_{24} \quad v_{35} \quad v_{36} \quad v_{45} \quad v_{46}]^T$$

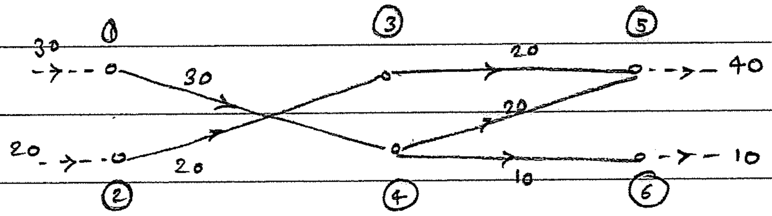
$$(= [x_{11} \quad x_{12} \quad x_{21} \quad x_{22} \quad z_{11} \quad z_{12} \quad z_{21} \quad z_{22}]^T),$$

$$c = [5 \quad 2 \quad 3 \quad 2 \quad 5 \quad 5 \quad 7 \quad 6]^T,$$

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 30 \\ 20 \\ 0 \\ 0 \\ -40 \end{bmatrix},$$

where we have ignored the redundant balance equation for the last node.

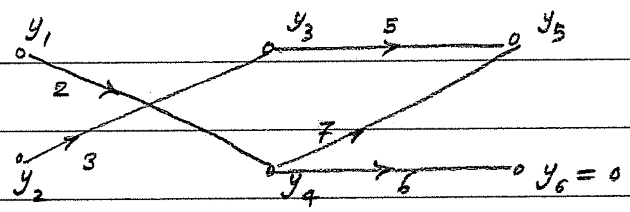
The proposed solution corresponds to a spanning tree (and hence is is a basic solution):



We now compute the vector y using

$$y_i - y_j = c_{ij} \quad \text{for all } (i,j) \in T,$$

and using $y_6 = 0$. (Here T denotes the spanning tree above.)



So we have

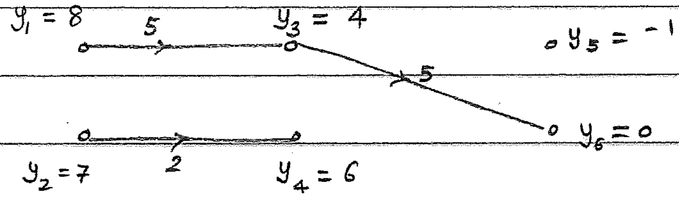
$$y_1 - y_4 = 2$$
$$y_2 - y_3 = 3$$
$$y_3 - y_5 = 5$$
$$y_4 - y_5 = 7$$
$$y_4 - \underbrace{y_6}_{0} = 6$$

So

$$y_4 = 6$$
$$y_5 = -1$$
$$y_3 = 4$$
$$y_2 = 7$$
$$y_1 = 8.$$

Next we compute the reduced costs for the non basic variables using

$$r_{ij} = c_{ij} - (y_i - y_j) \quad \text{for all } (i,j) \notin T.$$



So we have

$$r_{13} = 5 - (8 - 4) = 1 \geq 0,$$

$$r_{24} = 2 - (7 - 6) = 1 \geq 0,$$

$$r_{36} = 5 - (4 - 0) = 1 \geq 0.$$

Since $r_{ij} \geq 0$ for all $(i,j) \notin T$, the proposed solution is indeed optimal.

The optimal cost is given by

$$2 \cdot 30 + 3 \cdot 20 + 5 \cdot 20 + 7 \cdot 20 + 6 \cdot 10$$

$$= 60 + 60 + 100 + 140 + 60$$

$$= 420.$$

Exercise 7.6

(1) The incidence matrix is given by:

$$A = \begin{array}{c|ccccccc} \text{edge} & (1,2) & (1,5) & (2,3) & (2,5) & (3,4) & (5,3) & (5,4) \\ \hline \text{node } 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 4 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 5 & 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{array}$$

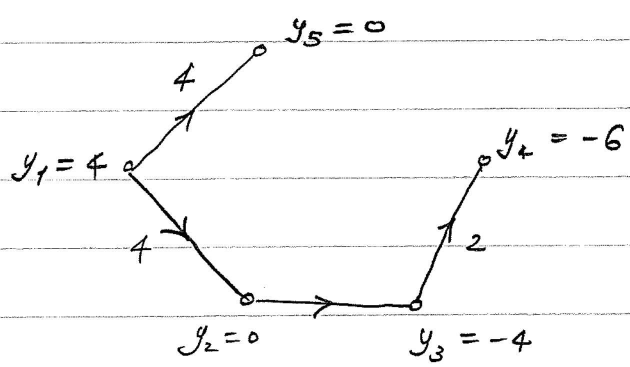
The constraints are given by

$$\begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

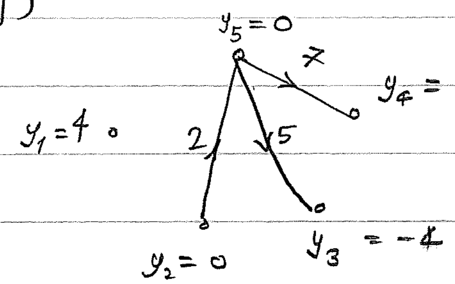
where $x = [x_{12} \quad x_{15} \quad x_{23} \quad x_{25} \quad x_{34} \quad x_{53} \quad x_{54}]^T$ and
$b = [5 \quad 5 \quad -4 \quad -3 \quad -3]^T$.

(2) The given solution satisfies the flow balance at each node, the flow in each edge is ≥ 0 and the nonzero flows are in edges which form a tree. So it is a basic feasible solution.

The simplex multipliers vector y can be determined using $c_{ij}' = y_i - y_j$ for tree edges (i,j).



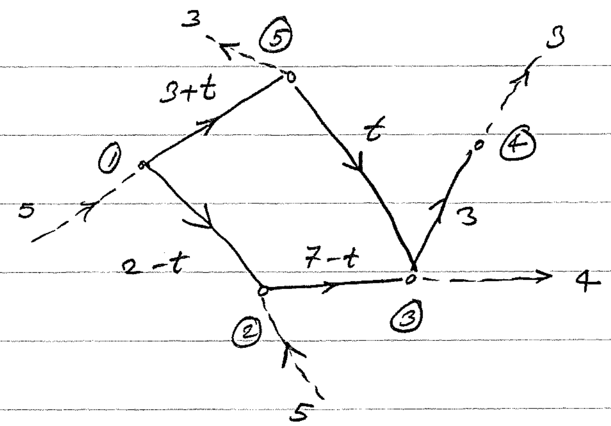
The reduced costs for the non-basic variables can be found out using $r_{ij} = c_{ij} - (y_i - y_j)$ for nontree edges (i,j)



$$r_{54} = c_{54} - (y_5 - y_4) = 7 - (0 - (-6)) = 1 \geqslant 0$$
$$r_{53} = c_{53} - (y_5 - y_3) = 5 - (0 - (-4)) = 1 \geqslant 0$$
$$r_{25} = c_{25} - (y_2 - y_5) = 2 - (0 - 0) = 2 \geqslant 0 .$$

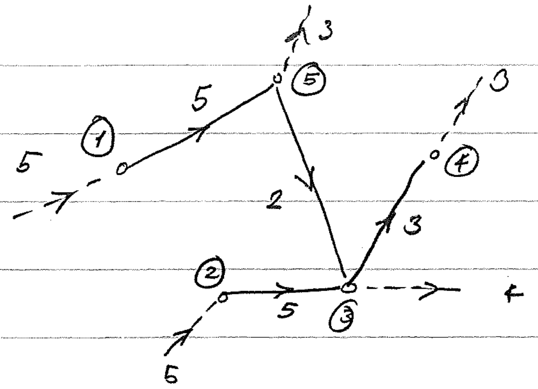
Since all $r_{ij} \geqslant 0$, we conclude that this (basic feasible) solution is optimal.

(3) Since only the cost vector has changed, the solution is still a basic feasible solution. Also the cost of a <u>nontree</u> edge has changed, and so the simplex multipliers vector is the same $(A_\beta^T y = c_\beta$; A_β, c_β are the same!). Also the reduced costs r_{54}, r_{25} are the same as before. We have now $r_{53} = c_{53} - (y_5 - y_3) = \boxed{3} - (0 - (-4)) = -1 < 0$. So the solution is not optimal. We let $x_{53} = t$ and let t increase from 0.

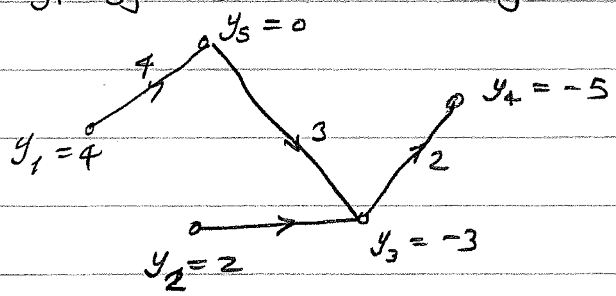


So t can increase up to a maximum of 2.

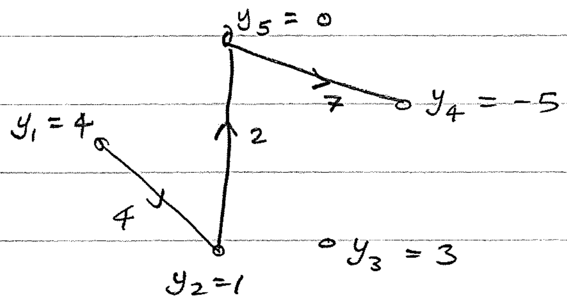
The new basic feasible solution is



The simplex multipliers vector y can be determined using $c_{ij} = y_i - y_j$ for tree edges (i,j):



The reduced costs for the nonbasic variables are given by $r_{ij} = c_{ij} - (y_i - y_j)$ for the nontree edges (i,j):



$r_{12} = c_{12} - (y_1 - y_2) = 4 - (4 - 1) = 1 \geq 0$

$r_{25} = c_{25} - (y_2 - y_5) = 2 - (1 - 0) = 1 \geq 0$

$r_{54} = c_{54} - (y_5 - y_4) = 7 - (0 - (-5)) = 2 \geq 0.$

Since all $r_{ij} \geq 0$, the new basic feasible solution is optimal.