Lösningar till tentan i SF1861 Optimeringslära, 18 Aug 2014

1.(a)

The considered LP problem is a minimum cost network flow problem with three nodes: 1, 2 and 3, and six arcs: (1,2), (2,1), (1,3), (3,1), (2,3) and (3,2).

The suggested solution $\hat{\mathbf{x}} = (15, 0, 10, 0, 0, 0)^\mathsf{T}$ corresponds to a spanning tree with the arcs (1,2) and (1,3), i.e. a basic solution. It is a feasible basic solution since all the balance equations (in all nodes) are satisfied and all variables are non-negative.

The simplex variables y_i are obtained from the equations $y_i - y_j = c_{ij}$ for basic variables, together with $y_3 = 0$. This gives
$y_3 = 0$,
$y_1 = y_3 + c_{13} = 0 + 2 = 2$,
$y_2 = y_1 - c_{12} = 2 - 3 = -1$.

Then the reduced costs for the non-basic variables are obtained from $r_{ij} = c_{ij} - y_i + y_j$:
$r_{21} = 1 - (-1) + 2 = 4$,
$r_{31} = 1 - 0 + 2 = 3$,
$r_{23} = 1 - (-1) + 0 = 2$,
$r_{32} = 1 - 0 + (-1) = 0$.

Since all $r_{ij} \geq 0$, the suggested solution is optimal.

However, since $r_{32} = 0$, the objective value will not change if we let x_{32} become a new basic variable. Let $x_{32} = t$ and increase t from 0. Then the basic variables will change according to $x_{12} = 15 - t$ and $x_{13} = 10 + t$.
In particular, with $t = 15$, we obtain a new optimal basic solution $\tilde{\mathbf{x}} = (0, 0, 25, 0, 0, 15)^\mathsf{T}$.

Check: $\mathbf{c}^\mathsf{T}\hat{\mathbf{x}} = 3 \cdot 15 + 2 \cdot 10 = 65$. $\mathbf{c}^\mathsf{T}\tilde{\mathbf{x}} = 2 \cdot 25 + 1 \cdot 15 = 65$.

1.(b)

We apply Gauss–Jordan's method on the given matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \\ 64 & 128 & 256 \end{bmatrix}$.

Add -8 times the first row to the second row and -64 times the first row to the third row.

Then the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is obtained, and $\mathbf{B}$ has been transformed to *reduced row echelon form* with only one *leading one*: $\mathbf{U} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$.

Note that $\mathcal{N}(\mathbf{B}^{\mathsf{T}})^{\perp} = \mathcal{R}(\mathbf{B})$, and that a basis to $\mathcal{R}(\mathbf{B})$ is obtained by chosing the columns in $\mathbf{B}$ corresponding to the "leading ones" in $\mathbf{U}$, i.e. the first column in $\mathbf{B}$.

Thus, the single vector $\begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}$ forms a basis to $\mathcal{R}(\mathbf{B})$, and thus also to $\mathcal{N}(\mathbf{B}^{\mathsf{T}})^{\perp}$.

In order to find a basis for $\mathcal{N}(\mathbf{B})$, note that the system $\mathbf{Bx} = \mathbf{0}$ is equivalent to the system $\mathbf{Ux} = \mathbf{0}$, i.e. $x_1 + 2x_2 + 4x_3 = 0$, for which the general solution is obtained by letting $x_2 = t$ and $x_3 = s$, where t and s are arbitrary real numbers. Then $x_1 = -2t - 4s$, and the general solution to $\mathbf{Bx} = \mathbf{0}$ can thus be written $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.

It follows that the two vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathcal{N}(\mathbf{B})$.

By repeating the above steps on $\mathbf{B}^{\mathsf{T}}$ instead of $\mathbf{B}$ the following is obtained:

The single vector $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ forms a basis to $\mathcal{R}(\mathbf{B}^{\mathsf{T}})$, and thus also to $\mathcal{N}(\mathbf{B})^{\perp}$.

The two vectors $\begin{pmatrix} -8 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -64 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $\mathcal{N}(\mathbf{B}^{\mathsf{T}})$.

Check of orthogonality:

$$\begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -8 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 8 \\ 64 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -64 \\ 0 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = 0.$$

The vector $(1, b, 1)^{\mathsf{T}}$ belong to $\mathcal{N}(\mathbf{B})$ if and only if $[\,1\ 2\ 3\,](1, b, 1)^{\mathsf{T}} = 0$,
i.e. if and only if $1 + 2b + 4 = 0$, i.e. if and only $b = -5/2$.

2.(a) Introduce the following new non-negative variables x'_j:
$x'_1 = x_1,\ \ x'_2 = x_2,\ \ x'_3 - x'_4 = x_3,$
$x'_5 =$ slack variable for the constraint $x_1 - x_2 + x_3 \geq 0$,
$x'_6 =$ slack variable for the constraint $x_2 + x_3 \geq 0$.
Further, introduce the variable vector $\mathbf{x}' = (x'_1, x'_2, x'_3, x'_4, x'_5, x'_6)^\mathsf{T}$.

Then the problem can be written as the following LP problem on standard form:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x}' \\ \text{subject to} \quad & \mathbf{A}\mathbf{x}' = \mathbf{b}, \quad \mathbf{x}' \geq \mathbf{0}, \end{aligned}$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{c} = (0,0,1,-1,0,0)^\mathsf{T}.$$

2.(b) and **2.(c)**
The suggested solution $\hat{\mathbf{x}} = (2,\, 1, -1)^\mathsf{T}$ corresponds to the solution $\hat{\mathbf{x}}' = (2,\, 1, 0, 1, 0, 0)^\mathsf{T}$ to the above problem on standard form. The optimality of this solution can be verified by showing that $\hat{\mathbf{x}}'$ is a feasible basic solution with non-negative reduced costs.
Alternatively, the optimality of $\hat{\mathbf{x}} = (2,\, 1, -1)^\mathsf{T}$ can be verified using the complementarity theorem. This is the approach used here, and then 2.(c) is simultaneously solved.
When the primal problem is

$$\begin{aligned} \text{P:}\ \text{minimize} \quad & x_3 \\ \text{subject to} \quad & x_1 - x_2 + x_3 \geq 0, \\ & x_2 + x_3 \geq 0, \\ & x_1 + x_2 = 3, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \text{ "free"}, \end{aligned}$$

the corresponding dual problem is

$$\begin{aligned} \text{D:}\ \text{maximize} \quad & 3y_3 \\ \text{subject to} \quad & y_1 + y_3 \leq 0, \\ & -y_1 + y_2 + y_3 \leq 0, \\ & y_1 + y_2 = 1, \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \text{ "free"}. \end{aligned}$$

The complementary theorem says that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal solutions to P and D, respectively, if and only if

(1) $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are feasible solutions to P and D,

(2) $\hat{y}_1(\hat{x}_1 - \hat{x}_2 + \hat{x}_3) = 0,\ \hat{y}_2(\hat{x}_2 + \hat{x}_3) = 0,\ \hat{x}_1(\hat{y}_1 + \hat{y}_3) = 0$ and $\hat{x}_2(-\hat{y}_1 + \hat{y}_2 + \hat{y}_3) = 0$.

Since the suggested point $\hat{\mathbf{x}} = (2,\, 1, -1)^\mathsf{T}$ is a feasible solution to P, it is an optimal solution to P if and only if there is a feasible solution $\hat{\mathbf{y}}$ to D such that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ satisfy (2) above.

Note that $\hat{\mathbf{x}}$ satisfies $\hat{x}_1 - \hat{x}_2 + \hat{x}_3 = 0,\ \hat{x}_2 + \hat{x}_3 = 0,\ \hat{x}_1 + \hat{x}_2 = 3,\ \hat{x}_1 > 0$ and $\hat{x}_2 > 0$.
Thus, $\hat{\mathbf{y}}$ must satisfy $\hat{y}_1 + \hat{y}_3 = 0,\ -\hat{y}_1 + \hat{y}_2 + \hat{y}_3 = 0,\ \hat{y}_1 + \hat{y}_2 = 1,\ \hat{y}_1 \geq 0$ and $\hat{y}_2 \geq 0$.
The unique solution to the first three equations is $\hat{\mathbf{y}} = (1/3,\, 2/3,\, -1/3)^\mathsf{T}$, and since this solution satisfies $\hat{y}_1 \geq 0$ and $\hat{y}_2 \geq 0$, it follows that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ satisfy (1) and (2) above.

Thus, $\hat{\mathbf{x}} = (2,\, 1, -1)^\mathsf{T}$ and $\hat{\mathbf{y}} = (1/3,\, 2/3,\, -1/3)^\mathsf{T}$ are optimal solutions to P and D.
The optimal value of P $= \hat{x}_3 = -1$. The optimal value of D $= 3\hat{y}_3 = 3 \cdot (-1/3) = -1$.

3.(a)

The objective function is $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$, with $\mathbf{H} = \begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}$, $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$.

LDL^T-factorization of $\mathbf{H}$ gives

$$\mathbf{H} = \mathbf{L}\mathbf{D}\mathbf{L}^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ -1.5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2.5 & 0 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & -1.5 & -1.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is a negative diagonal element in $\mathbf{D}$, the matrix $\mathbf{H}$ is *not* positive semidefinite, which in turn implies that there is no optimal solution to the problemen of minimizing $f(\mathbf{x})$ without constraints. (With e.g. $\mathbf{d} = (1,1,1)^\mathsf{T}$, $f(t\,\mathbf{d}) = -12t^2 + 60t \to -\infty$ when $t \to \infty$.)

3.(b)

We now have a QP problem with equality constraints, i.e. a problem of the form

minimize $\frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$,

where $\mathbf{A} = [\,1\ \ 1\ \ 1\,]$, $\mathbf{b} = 3$, $\mathbf{H} = \begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}$ and $\mathbf{c} = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$.

The general solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e. to $x_1 + x_2 + x_3 = 3$, is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot v_1 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot v_2,$$ for arbitrary values on v_1 and v_2,

which means that $\bar{\mathbf{x}} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ is a feasible solution, and $\mathbf{Z} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a matrix whos columns form a basis for the null space of $\mathbf{A}$.

After the variable change $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{Z}\mathbf{v}$ we should solve the system $(\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z})\mathbf{v} = -\mathbf{Z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$, provided that $\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z}$ is at least positive semidefinite.

We have $\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z} = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$, which is positive definite (since $10 > 0$, $10 > 0$, $10 \cdot 10 - 5 \cdot 5 > 0$).

The system $(\mathbf{Z}^\mathsf{T}\mathbf{H}\mathbf{Z})\mathbf{v} = -\mathbf{Z}^\mathsf{T}(\mathbf{H}\bar{\mathbf{x}} + \mathbf{c})$ becomes

$\begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$, with the unique solution $\hat{\mathbf{v}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which implies that

$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{Z}\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ is the unique optimal solution to our problem.

4.(a)
The objective function is $f(\mathbf{x}) = (x_1^2 + x_2^2 + 1)^{1/2} - 0.3\,x_1 - 0.4\,x_2$.

The gradient of f becomes $\nabla f(\mathbf{x}) = \left(\dfrac{x_1}{(x_1^2+x_2^2+1)^{1/2}} - 0.3,\ \dfrac{x_2}{(x_1^2+x_2^2+1)^{1/2}} - 0.4 \right)$.

The Hessian of f becomes $\mathbf{F}(\mathbf{x}) = \dfrac{1}{(x_1^2+x_2^2+1)^{3/2}} \cdot \begin{bmatrix} 1+x_2^2 & -x_1x_2 \\ -x_1x_2 & 1+x_1^2 \end{bmatrix}$.

The starting point is given by $\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and then

$f(\mathbf{x}^{(1)}) = 1$, $\nabla f(\mathbf{x}^{(1)}) = (-0.3, -0.4)$ and $\mathbf{F}(\mathbf{x}^{(1)}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since a diagonal matrix with strictly positive diagonal elements is positive definite, the Hessian $\mathbf{F}(\mathbf{x}^{(1)})$ is positive definite, and then the first Newton search direction $\mathbf{d}^{(1)}$ is obtained by solving the system

$\mathbf{F}(\mathbf{x}^{(1)})\mathbf{d} = -\nabla f(\mathbf{x}^{(1)})^\mathsf{T}$, i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{d} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$, with the solution $\mathbf{d}^{(1)} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$.

First try $t_1 = 1$, so that $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + t_1\mathbf{d}^{(1)} = \mathbf{x}^{(1)} + \mathbf{d}^{(1)} = \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix}$.

Then $f(\mathbf{x}^{(2)}) = \sqrt{1.25} - 0.09 - 0.16 < 1.2 - 0.25 < 1 = f(\mathbf{x}^{(1)})$,
so $t_1 = 1$ is accepted, and the first iteration is completed.

4.(b)
The function f is convex on $I\!R^2$ if and only if the Hessian

$\mathbf{F}(\mathbf{x}) = \dfrac{1}{(x_1^2+x_2^2+1)^{3/2}} \cdot \begin{bmatrix} 1+x_2^2 & -x_1x_2 \\ -x_1x_2 & 1+x_1^2 \end{bmatrix}$ is positive semidefinite for all $\mathbf{x} \in I\!R^2$,

which holds if and only if $\begin{bmatrix} 1+x_2^2 & -x_1x_2 \\ -x_1x_2 & 1+x_1^2 \end{bmatrix}$ is positive semidefinite for all $\mathbf{x} \in I\!R^2$.

But $1+x_2^2 > 0$, $1+x_1^2 > 0$, and $(1+x_2^2)(1+x_1^2) - (-x_1x_2)(-x_1x_2) = 1 + x_1^2 + x_2^2 > 0$

for all $\mathbf{x} \in I\!R^2$, which implies that $\mathbf{F}(\mathbf{x})$ is in fact positive definite for all $\mathbf{x} \in I\!R^2$, which in turn implies that f is strictly convex on the whole set $I\!R^2$.

4.(c)
We should solve $\nabla f(\mathbf{x}) = (0,0)$, i.e. $\dfrac{x_1}{(x_1^2+x_2^2+1)^{1/2}} = 0.3$ and $\dfrac{x_2}{(x_1^2+x_2^2+1)^{1/2}} = 0.4$.

Some analytical calculations show that the only solution to this system is

$\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)^\mathsf{T} = \left(\dfrac{0.6}{\sqrt{3}}, \dfrac{0.8}{\sqrt{3}} \right)^\mathsf{T}$.

Since f is strictly convex on $I\!R^2$, $\hat{\mathbf{x}}$ is the unique globally optimal solution to the problem of minimizing $f(\mathbf{x})$ on $I\!R^2$.

5.(a)

Lagrangefunktionen kan skrivas $L(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) + y_1 g_1(\mathbf{x}) + y_2 g_2(\mathbf{x}) =$

$$\frac{1}{1-x_1} + \frac{4}{1-x_2} + \frac{9}{1-x_3} + \frac{y_1}{1+x_1} + \frac{y_1}{1+x_2} + \frac{y_1}{1+x_3} - 3y_1 + y_2x_1^2 + y_2x_2^2 + y_2x_3^2 - 0.5y_2.$$

KKT-villkoren kan delas upp i fyra grupper enligt följande.

(KKT–1) $\partial L/\partial x_j = 0$ för $j = 1,2,3$:

$$\frac{1}{(1-x_1)^2} - \frac{y_1}{(1+x_1)^2} + 2y_2x_1 = 0,$$
$$\frac{4}{(1-x_2)^2} - \frac{y_1}{(1+x_2)^2} + 2y_2x_2 = 0,$$
$$\frac{9}{(1-x_3)^2} - \frac{y_1}{(1+x_3)^2} + 2y_2x_3 = 0.$$

(KKT–2) Tillåten punkt, dvs $g_i(\mathbf{x}) \leq 0$ för $i = 1,2$:

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} - 3 \leq 0,$$
$$x_1^2 + x_2^2 + x_3^2 - 0.5 \leq 0.$$

(KKT–3) Lagrangemultiplikatorerna icke-negativa:

$y_1 \geq 0,$
$y_2 \geq 0.$

(KKT–4) Komplementaritetsvillkor, dvs $y_i g_i(\mathbf{x}) = 0$ för $i = 1,2$:

$$y_1\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} - 3\right) = 0,$$
$$y_2\,(x_1^2 + x_2^2 + x_3^2 - 0.5) = 0.$$

5.(b)

Vi söker en lösning $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ som dels uppfyller ovanstående KKT-villkor,
dels uppfyller att $g_1(\hat{\mathbf{x}}) = 0$ och $g_2(\hat{\mathbf{x}}) < 0$.

Men om $g_2(\hat{\mathbf{x}}) < 0$ så måste $\hat{y}_2 = 0$ enligt (KKT-4).
Då kan inte $\hat{y}_1 = 0$, ty om $\hat{y}_1 = \hat{y}_2 = 0$ så finns ingen lösning till (KKT-1).
Vi kan alltså förutsätta att $\hat{y}_1 > 0$ och $\hat{y}_2 = 0$.

Då ger (KKT-1) att $\hat{x}_1 = \dfrac{\sqrt{\hat{y}_1} - 1}{\sqrt{\hat{y}_1} + 1}$, $\hat{x}_2 = \dfrac{\sqrt{\hat{y}_1} - 2}{\sqrt{\hat{y}_1} + 2}$, $\hat{x}_3 = \dfrac{\sqrt{\hat{y}_1} - 3}{\sqrt{\hat{y}_1} + 3}$.

Vidare ger (KKT-4) att $\dfrac{1}{1+\hat{x}_1} + \dfrac{1}{1+\hat{x}_2} + \dfrac{1}{1+\hat{x}_3} = 3$, (eftersom $\hat{y}_1 > 0$).

Om vi kombinerar dessa två senaste rader så erhålls efter enkla kalkyler att

$\hat{y}_1 = 4$, $\hat{x}_1 = \dfrac{1}{3}$, $\hat{x}_2 = 0$, $\hat{x}_3 = -\dfrac{1}{5}$. Dessutom är enligt ovan $\hat{y}_2 = 0$.

Vi ser att $g_2(\hat{\mathbf{x}}) = 1/9 + 1/25 - 3 < 0$ som vi önskade.
Därmed har vi hittat den lösning vi sökte.

5.(c)

Notera att varje tillåten lösning till problemet uppfyller att $x_1^2 + x_2^2 + x_3^2 \leq 0.5$, vilket bland annat betyder att $-1 < x_j < 1$ för alla tillåtna lösningar $\mathbf{x}$.

Andraderivatsmatriserna till målfunktionen och bivillkorsfunktionerna är

$$\begin{bmatrix} \frac{2}{(1-x_1)^3} & 0 & 0 \\ 0 & \frac{8}{(1-x_2)^3} & 0 \\ 0 & 0 & \frac{18}{(1-x_3)^3} \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{(1+x_1)^3} & 0 & 0 \\ 0 & \frac{2}{(1+x_2)^3} & 0 \\ 0 & 0 & \frac{2}{(1+x_3)^3} \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Eftersom (enligt ovan) $-1 < x_j < 1$ för alla tillåtna lösningar så är samtliga ovanstående matriser positivt definita för alla tillåtna $\mathbf{x}$ (ty de är alla diagonalmatriser med strikt positiva diagonalelement). Därmed är såväl målfunktionen som bivillkorsfunktionerna (strikt) konvexa, vilket enligt en känd sats medför att varje KKT-punkt är en globalt optimal lösning till problemet

$\hat{\mathbf{x}}$ från (b)-uppgiften är alltså en globalt optimal lösning.

5.(d)

Eftersom (enligt ovan) det studerade problemet är konvext med *strikt* konvex målfunktion så kan det inte finnas flera olika optimala lösningar.
Därför är $\hat{\mathbf{x}}$ från (b)-uppgiften den enda globalt optimala lösningen.
Då kan det inte finnas någon KKT-lösning $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ med $\tilde{\mathbf{x}} \neq \hat{\mathbf{x}}$, ty i såfall skulle även $\tilde{\mathbf{x}}$ vara en globalt optimal lösning, vilket vi nyss konstaterat är omöjligt.
Slutligen kan det inte finnas någon KKT-lösning $(\hat{\mathbf{x}}, \tilde{\mathbf{y}})$ med $\tilde{\mathbf{y}} \neq \hat{\mathbf{y}}$, ty om $(\hat{\mathbf{x}}, \tilde{\mathbf{y}})$ är en KKT-lösning så måste $\tilde{y}_2 = 0$ (ty $g_2(\hat{\mathbf{x}}) < 0$) och då måste enligt (KKT-1) $\tilde{y}_1 = 4$.