KTH Mathematics

SF2812 Applied linear optimization, final exam
Tuesday October 23 2007 14.00–19.00
Brief solutions

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1. (a) From the GAMS output file, the values of "`VAR x`" suggest $x = (5/3\ 0\ 0\ 13/3\ 4/3)$, the marginal costs for "`EQU cons`" suggest $y = (1/3\ 0\ 0)^T$, and the marginal costs for "`VAR x`" suggest $s = (0\ 5/3\ 5/3\ 0\ 0)^T$. We have $Ax = b$, $A^Ty + s = c$, $x \geq 0$, $s \geq 0$ and $x^Ts = 0$. Hence, the solutions are optimal to the respective problem.

(b) Since $s_2 = s_3 = 5/3$, it follows that the optimal solution is unchanged as long as the costs of x_2 or x_3 are not decreased more than $5/3$. Hence, the solution is not at all sensitive to changes considered by AF. The computed optimal solution is optimal also considering the fluctuations.

(c) Since $y_1 = 1/3$, the optimal value is expected to change with $1/3$ per unit change of b_1.

2. (a) Since $x(\mu)$ and $y(\mu)$ that are solution and Lagrange multipliers to (P_μ) also solve the primal-dual nonlinear equations, we immediately obtain

$$x(\mu) \approx \begin{pmatrix} 0.0008 \\ 2.9614 \\ 3.0185 \\ 1.0006 \\ 0.0199 \\ 0.0010 \end{pmatrix}, \quad y(\mu) \approx \begin{pmatrix} 0.2502 \\ -0.2003 \\ 0.7497 \end{pmatrix}.$$

Finally, we may obtain $s(\mu)$ from $s_j(\mu) = \mu/x_j(\mu)$, $j = 1, \dots, 6$, and it follows that

$$s(\mu) \approx \begin{pmatrix} 1.2992 \\ 0.0003 \\ 0.0003 \\ 0.0010 \\ 0.0503 \\ 1.0000 \end{pmatrix}.$$

(b) Since we expect the solutions to be in the order of 10^{-3} away from an optimal solution, rounding gives

$$x = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{5} \\ \frac{3}{4} \end{pmatrix}.$$

We may then compute

$$s = c - A^Ty = \begin{pmatrix} \frac{13}{10} \\ 0 \\ 0 \\ 0 \\ \frac{1}{20} \\ 1 \end{pmatrix}.$$

We have $Ax = b$, $A^Ty + s = c$, $x \geq 0$, $s \geq 0$ and $x^Ts = 0$. Hence, the solutions are optimal to the respective problem.

(c) The computed solution is a basic feasible solution. In addition, since strict complementarity holds, the solution is unique. Consequently, the simplex method would compute the same solution.

3. (See the course material.)

4. (a) For a fix vector $u \in I\!R^n$, Lagrangian relaxation of the first group of constraints gives

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{j=1}^n f_j z_j - \sum_{i=1}^n u_i \left(\sum_{j=1}^n x_{ij} - 1 \right) \\ \text{subject to} & \sum_{i=1}^n a_i x_{ij} \geq b_j z_j, \quad j = 1, \dots, n, \\ & x_{ij} \in \{0,1\},\ z_j \in \{0,1\}, \quad i = 1, \dots, n,\ j = 1, \dots, n, \end{array}$$

This problem decomposes into one problem for each j as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n (c_{ij} - u_i) x_{ij} - f_j z_j \\ \text{subject to} & \sum_{i=1}^n a_i x_{ij} \geq b_j z_j, \\ & x_{ij} \in \{0,1\},\ z_j \in \{0,1\}, \quad i = 1, \dots, n, \end{array}$$

for $j = 1, \dots, n$. For each j, we may solve two problem by equating $z_j = 0$ and $z_j = 1$ respectively. For $z_j = 0$ we obtain $x_{ij} = 0$ or $x_{ij} = 1$ depending on whether $c_{ij} - u_i$ is positive or negative. For $z_j = 1$ we obtain a "knapsack-like" problem in the x_{ij}-variables.

(b) For a fix nonnegative vector $v \in I\!R^m$, Lagrangian relaxation of the second group of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{j=1}^n f_j z_j - \sum_{j=1}^n v_j \left(\sum_{i=1}^n a_i x_{ij} - b_j z_j \right) \\ \text{subject to} & \displaystyle\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \\ & x_{ij} \in \{0,1\}, \; z_j \in \{0,1\}, \quad i = 1, \dots, n, \; j = 1, \dots, n, \end{array}$$

This problem separates into two separate problems in the x_{ij}-variables and the z_j-variables respectively. The problem in the x_{ij}-variables decomposes into trivial problems for each i according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^n (c_{ij} - a_i v_j) x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^n x_{ij} = 1, \\ & x_{ij} \in \{0,1\}, \quad j = 1, \dots, n, \end{array}$$

for $i = 1, \dots, n$. These can be solved directly by noting which x_{ij}-variable that has the smallest coefficient in the objective function. Similarly, the problem in the z_j-variables decomposes into trivial problems for each j according to

$$\begin{array}{ll} \text{minimize} & (b_j v_j - f_j) z_j \\ \text{subject to} & z_j \in \{0,1\}, \end{array}$$

for $j = 1, \dots, n$. Here, $z_j = 0$ or $z_j = 1$ depending on whether $b_j v_j - f_j$ is positive or negative.

(c) The second relaxation gives a relaxed problem with integer optimal solutions even if the integrality requirement is relaxed. Hence, the corresponding dual underestimate become identical with the one obtained from an LP-relaxation.

The first relaxation gives a more complicated relaxed problem, where the integrality requirement is essential, in general. Hence, one would here expect the underestimate to be better than what the LP-relaxation would give. (We know that it is always at least as good.)

5. The suggested initial extreme points $v_1 = (1\ 0\ 0\ 1)^T$ and $v_2 = (-1\ 0\ 0\ 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} 8 & 2 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (6\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (10\ -4)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ -\frac{26}{3} \end{pmatrix}.$$

By forming $c - y_1 A = (0\ 4/3\ -13/3\ -26/3)$ we obtain the subproblem

$$\begin{array}{lll} \frac{26}{3} + \frac{1}{3} & \text{minimize} & 4x_2 - 13x_3 - 26x_4 \\ & \text{subject to} & -1 \le x_1 + x_2 \le 1, \\ & & -1 \le x_1 - x_2 \le 1, \\ & & -1 \le x_3 + x_4 \le 1, \\ & & -1 \le x_3 - x_4 \le 1. \end{array}$$

The resulting optimal solution gives a new extreme point $v_3 = (0\ -1\ 0\ 1)^T$ with reduced cost $-4/3$. The corresponding column in the master problem is $(3\ 1)^T$, and we obtain

$$p_B = -B^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = -\begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

By considering the step from α_B along p_B an requiring nonnegativity, we obtain the maximum steplength as $2/5$, and α_2 leaves the basis. Hence, α_3 replaces α_2 as basic variable.

The basic variables are now given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{2}{5} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (10\ -3)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{13}{5} \\ -\frac{54}{5} \end{pmatrix}.$$

By forming $c - y_1 A = (-4/5\ \ 4/5\ -27/5\ -10)$ we obtain the subproblem

$$\begin{array}{lll} \frac{54}{5} + \frac{1}{5} & \text{minimize} & -4x_1 + 4x_2 - 27x_3 - 50x_4 \\ & \text{subject to} & -1 \le x_1 + x_2 \le 1, \\ & & -1 \le x_1 - x_2 \le 1, \\ & & -1 \le x_3 + x_4 \le 1, \\ & & -1 \le x_3 - x_4 \le 1. \end{array}$$

The resulting optimal solutions are v_1 and v_3, which both give reduced cost 0. Hence, we have found an optimal solution to the original problem. The solution is given by

$$v_1 \alpha_1 + v_3 \alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{3}{5} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \frac{2}{5} = \begin{pmatrix} \frac{3}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}.$$