KTH Mathematics

SF2812 Applied linear optimization, final exam
Friday January 15 2010 8.00–13.00
Brief solutions

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1. (a) The primal-dual system of equations can be written as

$$x_1 + 3x_2 = 1, \tag{1a}$$
$$y + s_1 = 1, \tag{1b}$$
$$3y + s_2 = 2, \tag{1c}$$
$$x_1 s_1 = \mu, \tag{1d}$$
$$x_2 s_2 = \mu. \tag{1e}$$

where we also implicitly require $x > 0$ and $s > 0$. We may eliminate y from (1b) and (1c), which gives $3s_1 - s_2 = 1$. By (1d) and (1e) we may express $s_1 = \mu/x_1$ and $s_2 = \mu/x_2$, so that this equation takes the form $3\mu/x_1 - \mu/x_2 = 1$. Hence,

$$3\mu x_2 - \mu x_1 - x_1 x_2 = 0. \tag{2}$$

By (1a), $x_1 = 1 - 3x_2$, which inserted into (2) gives

$$x_2^2 + \left(2\mu - \frac{1}{3}\right) x_2 - \frac{\mu}{3} = 0. \tag{3}$$

By taking the positive root of (3), we obtain

$$x_2(\mu) = \frac{1}{6} - \mu + \sqrt{\mu^2 + \frac{1}{36}}.$$

Hence,

$$\begin{aligned}
x_1(\mu) &= 1 - 3x_2(\mu) = \ldots = \frac{1}{2} + 3\mu - 3\sqrt{\mu^2 + \frac{1}{36}},\\
x_2(\mu) &= \frac{1}{6} - \mu + \sqrt{\mu^2 + \frac{1}{36}},\\
s_1(\mu) &= \frac{\mu}{x_1(\mu)} = \ldots = \frac{1}{6} + \mu + \sqrt{\mu^2 + \frac{1}{36}},\\
s_2(\mu) &= \frac{\mu}{x_2(\mu)} = \ldots = \frac{1}{2} - 3\mu - 3\sqrt{\mu^2 + \frac{1}{36}},\\
y(\mu) &= 1 - s_1(\mu) = \ldots = \frac{5}{6} - \mu - \sqrt{\mu^2 + \frac{1}{36}}.
\end{aligned}$$

(b) Letting $\mu \to 0$ gives

$$x = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}, \quad y = \frac{2}{3}, \quad s = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}.$$

It is straightforward to very that $Ax = b$, $x \ge 0$, $A^T y + s = c$, $s \ge 0$, $x^T s = 0$. Consequently, optimality holds.

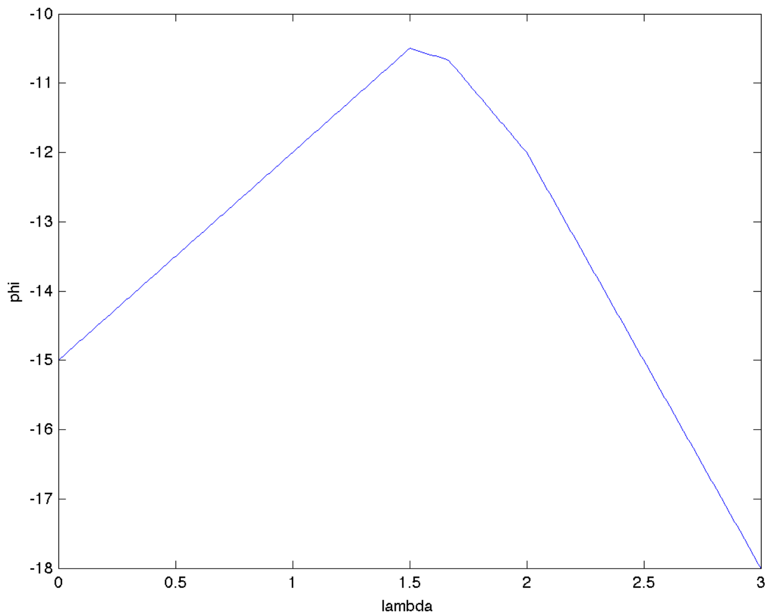
2. (a) We get

$$\varphi(\lambda) = -\lambda b + \sum_{j=1}^{n} \min_{x_j \in \{0,1\}} (a_j\lambda - c_j)x_j = -\lambda b - \sum_{j=1}^{n} (c_j - \lambda a_j)_+,$$

(b) For a given λ, an optimal solution to the Lagrangian relaxed problem is $x(\lambda)$, with $x_j(\lambda) = 1$ for j such that $\lambda a_j - c_j < 0$ and $x_j(\lambda) = 0$ for j such that $\lambda a_j - c_j \geq 0$. A subgradient is now given by

$$-\left(\sum_{j=1}^{n} a_j x_j(\lambda) + b\right) = -b + \sum_{j: a_j\lambda < c_j} a_j.$$

(c) The dual problem can be illustrated grahically as in the following figure.



The figure gives $\lambda^* = 1.5$ with $\varphi(\lambda^*) = -10.5$.
By inspection one can see that the optimal solution to (KP) is given by $x^* = (1\ 0\ 1)^T$ with $c^T x^* = -9$.
The duality gap is therefore 1.5.

3. (See the course material.)

4. (a) It is straightforward to verify that $\widehat{X} \geq 0$, $\widehat{S} \geq 0$, $\sum_{j=1}^{3} \widehat{x}_{ij} = a_i$, $i = 1, 2, 3, 4$, $\sum_{i=1}^{4} x_{ij} = b_j$, $j = 1, 2, 3$, $u_i + v_j + s_{ij} = c_{ij}$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$. Consequently, $\widehat{X}$ is feasible to (PTP) and $\widehat{u}$, $\widehat{v}$ and $\widehat{S}$ are feasible to (DTP). In addition, complementarity holds since $\widehat{x}_{ij}\widehat{s}_{ij} = 0$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$. The solutions are thus optimal. In particular, $\widehat{X}$ is optimal to (PTP).

(b) We only expect an interior solver to give basic feasible solutions as optimal solutions when the solution is unique. This is because the barrier transformation makes the iterates tend to stay away from as many constraints as possible. The conclusion in this case is that the primal problem does not have a unique optimal solution.

(c) Since $\sum_{i=1}^{4} p_{ij} = 0$, $j = 1, 2, 3$, and $\sum_{j=1}^{3} p_{ij} = 0$, $i = 1, 2, 3, 4$, it follows that $\widehat{X} + \alpha P$ is feasible as long as $\widehat{X} + \alpha P \geq 0$. In addition, since $(\widehat{x}_{ij} + \alpha p_{ij})\widehat{s}_{ij} = 0$, $i = 1, 2, 3, 4$, $j = 1, 2, 3$ for all α, we conclude that $\widehat{X} + \alpha P \geq 0$ is optimal if it is feasible. Feasibility holds for $-1/2 \leq \alpha \leq 1/2$. By taking the limiting cases, we obtain

$$X_1 = \widehat{X} - \frac{1}{2}P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad X_2 = \widehat{X} + \frac{1}{2}P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Both X_1 and X_2 are integer optimal solutions.

(d) The positive components of X_1 and X_2 are uniquely determined by a and b. Hence, they both correspond to optimal basic feasible solutions. Consequently, X_1 and X_2 could both be given as optimal solutions by the simplex method.

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enters the basis.

The subproblem is given by

$$\begin{array}{rll} 1 - & \frac{1}{6}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 6\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 11, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (2\ 1\ 0)^T$ with optimal value $-1/6$. Hence, $a_4 = (2\ 1\ 0)^T$ and the maximum step is given by

$$0 \leq x = B^{-1}b - \eta B^{-1}a_4 = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence, $\eta_{\max} = 30$ and x_1 leaves the basis, so that the basic variables are given by $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$. The reduced costs are given by

$$y = B^{-T}e = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 1$.

The subproblem is given by

$$\begin{array}{rll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 11, \\ & & \alpha_i \ge 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$, with $a_2 = (0\ 2\ 0)^T$, $a_3 = (0\ 0\ 1)^T$ and $a_4 = (2\ 1\ 0)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 80 W-rolls, with 10 rolls cut according to pattern $(0\ 2\ 0)^T$, 40 rolls cut according to pattern $(0\ 0\ 1)^T$ and 30 rolls cut according to pattern $(2\ 1\ 0)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)