KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 20 2011 14.00–19.00
Brief solutions

---

1. (a) It approximately holds that $Ax = b$, $A^T y + s = c$ and $x_j s_j = 0$, $j = 1, 2, 3, 4$. However, $s_2 = -2 < 0$ contradicts optimality. In addition, x_2 is almost zero, but negative. One would expect all x-variables and s-variables to be positive in a solution provided by an interior method.

(b) It appears that the computed solution approximately satisfies $Ax = b$, $A^T y + s = c$ and $x_j s_j = \mu$, $j = 1, 2, 3, 4$, for a small value of μ. However, the positivity of x and s are not fulfilled. Therefore, I guess that AF has forgotten to limit the steplength in the Newton iterations so as to retain the positivity of x and s. More specifically, if Δx, Δy and Δs denote the Newton steps at a particular iteration, the steplength α must be chosen such that $x + \alpha\Delta x > 0$ and $s + \alpha\Delta s > 0$.

(c) As all optimality conditions except $s_2 \geq 0$ are approximately fulfilled, we obtain a near-optimal solution to the linear program where $x_2 \geq 0$ is replaced by $x_2 \leq 0$. To see this, note that the given x, y and s are near-optimal solutions to the primal-dual pairs of linear programs

$$(PLP') \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x_j \geq 0, \quad j = 1, 3, 4, \\ & x_2 \leq 0, \end{array}$$

$$(DLP') \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s_j \geq 0, \quad j = 1, 3, 4, \\ & s_2 \leq 0, \end{array}$$

so AF has in fact solved (PLP') instead of (PLP).

(d) Let $x_B = (x_1 \; x_4)^T$. Then B is nonsingular, so we have a basic feasible solution. Since $s_2 < 0$, x_2 will enter the basis. For $p_2 = 1$, we obtain the change in the basic variables from $Bp_B = -A_2$, i.e.,

$$\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

The maximum steplength is given by the maximum nonnegative α such that

$$x_B + \alpha p_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} + \alpha \begin{pmatrix} p_1 \\ p_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -2 \end{pmatrix} \geq 0,$$

i.e., $\alpha = 1$. Then x_4 becomes zero, so x_4 leaves the basis. The new basic feasible solution is given by $x_B = (x_1\ x_2)^T = (2\ 1)^T$. Compute y from $B^T y = c_B$ and let $s_N = c_N - N^T y$. We obtain

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \begin{pmatrix} s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $s \geq 0$, an optimal solution has been found. The optimal solution is $x = (2\ 1\ 0\ 0)^T$.

2. (See the course material.)

3. The suggested initial extreme points $v_1 = (-1\ \ -1\ \ -1\ \ 1)^T$ and $v_2 = (1\ \ -1\ \ -1\ \ 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (-3\ 1)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

By forming $c^T - y_1 A = (0\ \ 1\ \ 1\ -1)$ we obtain the subproblem

$$\begin{array}{lll} 3+ & \text{minimize} & x_2 + x_3 - x_4 \\ & \text{subject to} & -1 \leq x_j \leq 1, j = 1, \dots, 4. \end{array}$$

The given extreme points v_1 and v_2 are optimal to the subproblem, so the optimal value of the subproblem is zero, and the master problem has been solved. The solution to the original problem is given by

$$v_1\alpha_1 + v_3\alpha_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

4. (a) The dual objective $\varphi(u)$ is the optimal solution of

$$\begin{aligned}\varphi_1(u) = \min\ & -2x_1 - x_2 - 3x_3 - x_4 - u(1 - x_1 - x_3)\\ \text{s.t.}\ & 4x_1 + 5x_2 + 6x_3 + 7x_4 \le 10,\ x_j \in \{0,1\},\ j = 1,\dots,4,\\ = -u + \min\ & (u-2)x_1 - x_2 + (u-3)x_3 - x_4\\ \text{s.t.}\ & 4x_1 + 5x_2 + 6x_3 + 7x_4 \le 10,\ x_j \in \{0,1\},\ j = 1,\dots,4.\end{aligned}$$

(b) The feasible solutions to the Lagrangian relaxation problem and their corresponding values of $c^Tx - u(a^Tx - b)$ are given by

$$\begin{aligned}
x^{(1)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T, & c^Tx^{(1)} - u(a^Tx^{(1)} - b) &= -u\\
x^{(2)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, & c^Tx^{(2)} - u(a^Tx^{(2)} - b) &= -2\\
x^{(3)} &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, & c^Tx^{(3)} - u(a^Tx^{(3)} - b) &= -1 - u\\
x^{(4)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, & c^Tx^{(4)} - u(a^Tx^{(4)} - b) &= -3\\
x^{(5)} &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, & c^Tx^{(5)} - u(a^Tx^{(5)} - b) &= -1 - u\\
x^{(6)} &= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T, & c^Tx^{(6)} - u(a^Tx^{(6)} - b) &= -3\\
x^{(7)} &= \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T, & c^Tx^{(7)} - u(a^Tx^{(7)} - b) &= -5 + u.
\end{aligned}$$

By finding the minimum for each $u \ge 0$, it follows that $\varphi(u) = -5 + u$ if $0 \le u \le 2$ and $\varphi(u) = -1 - u$ if $u \ge 2$. Hence, $u^* = 2$.

(c) There are five optimal solutions to the Lagrangian relaxation problem for $u = 2$, namely $x^{(3)}$, $x^{(4)}$, $x^{(5)}$, $x^{(6)}$ and $x^{(7)}$. They give subgradients $-(a^Tx^{(k)} - b)$, $k = 3,4,5,6,7$, as -1, 0, -1, 0 and 1 respectively.

(As $-(a^Tx^{(k)} - b) = 0$ for $k = 4, 6$, it follows that $x^{(4)}$ and $x^{(6)}$ are optimal to (IP).)

5. (a) Since $c \ge 0$, a feasible solution to (DLP) is given by $y = 0$, $s = c$. Hence, $\text{optval}(DLP) \ge 0$ as $b^Ty = 0$ for this solution.

(b) Since $c \ge 0$ and $x \ge 0$ for all x that are feasible to (PLP), it holds that $c^Tx \ge 0$. Hence, $\text{optval}(PLP) \ge 0$. It may hold that (PLP) is infeasible, e.g., $A = -1$, $b = 1$.

(c) Since (DLP) is feasible, there is a feasible solution y, s. Then, $y + \alpha\eta$ is feasible for all $\alpha \ge 0$, since $A^T(y + \alpha\eta) \le A^Ty \le c$. Since $b^T(y + \alpha\eta)$ tends to infinity as $\alpha \to \infty$, we conclude that $\text{optval}(PLP) = \text{optval}(DLP) = \infty$.

(d) If there is no η such that $A^T\eta \le 0$ and $b^T\eta > 0$, then the primal-dual pairs of linear programs

$$(PLP')\quad \begin{array}{ll} \text{minimize} & 0^Tx \\ \text{subject to} & Ax = b,\\ & x \ge 0, \end{array} \qquad (DLP')\quad \begin{array}{ll} \text{maximize} & b^T\eta \\ \text{subject to} & A^T\eta \le 0, \end{array}$$

have optimal value zero. Hence, (PLP') is feasible. Thus, (PLP) is feasible, since (PLP') and (PLP) have the same feasible region. Consequently, (PLP) and (DLP) are both feasible, so they have the same finite optimal value by strong duality.