KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 18 2012 14.00–19.00
Brief solutions

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1. As $\widehat{x}_j > 0$, $j = 1, 2, 3, 4$, the active constraints at $\widehat{x}$ are given by

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \\ \widehat{x}_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 7 \\ 0 \end{pmatrix}.$$

These constraints remain active for $\widehat{x} + \alpha p$, where p satisfies

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the given hint we obtain $p = (2 \ \ -1 \ \ 3 \ \ -1 \ \ 0)^T$. The additional requirement $\widehat{x} + \alpha p \geq 0$ gives

$$\begin{pmatrix} 3 \\ 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $\widehat{x} + \alpha p \geq 0$ for $-1 \leq \alpha \leq 1$. In addition, it holds that $c^T p = 0$, so that $\widehat{x} + \alpha p$ has the same objective function value as $\widehat{x}$ for all α. By taking the limiting values of α, we get two new points at which five constraints are active, namely

$$x^{(1)} = \widehat{x} - p = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad x^{(2)} = \widehat{x} + p = \begin{pmatrix} 5 \\ 1 \\ 6 \\ 0 \\ 0 \end{pmatrix}.$$

As there are five active constraints at these points, we expect them to be basic feasible solutions. By assuming that x_1, x_2 and x_4 are basic variables, corresponding to $x^{(1)}$, we may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

with solution $y = (-1\ 0\ 1)^T$, so that $s = c - A^T y = (0\ 0\ 0\ 0\ 1)^T$. As $s \geq 0$, we have verified optimality of $x^{(1)}$, and hence $\widehat{x}$ and $x^{(2)}$ are optimal as well. It is straightforward to verify that $x^{(2)}$ is also a basic feasible solution at which x_1, x_2 and x_3 are basic variables.

2. (a) With $X = \operatorname{diag}(x)$ and $S = \operatorname{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix},$$

for a suitable value of the barrier parameter μ. We may for example let $\mu = x^T s / n = 5$. Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ -4 \\ -2 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) We would compute $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ as $x^{(1)} = x^{(0)} + \alpha \Delta x^{(0)}$, $y^{(1)} = y^{(0)} + \alpha \Delta y^{(0)}$, $s^{(1)} = s^{(0)} + \alpha \Delta s^{(0)}$, where α is a positive steplength. In a pure Newton step, $\alpha = 1$, but we must also maintain $x^{(1)} > 0$ and $s^{(1)} > 0$. We may compute $\alpha_{\max}$ as the largest step α for which $x + \alpha \Delta x \geq 0$ and $s + \alpha \Delta s \geq 0$. We may then let $\alpha = \min\{1, 0.99\alpha_{\max}\}$ to ensure positivity of $x^{(1)} > 0$ and $s^{(1)} > 0$. (In order to get a convergent method, some additional condition on α ensuring proximity to the barrier trajectory may need to be imposed.)

3. (See the course material.)

4. The suggested initial extreme points $v_1 = (1 \;\; 1 \;\; 1)^T$ and $v_2 = (-1 \;\; -1 \;\; -1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (0 \; 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1 \; c^T v_2) = (3 \;\; -3)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}.$$

By forming $c^T - y_1 A = (1/2 \;\; -8 \;\; 15/2)$ we obtain the subproblem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x_1 - 8x_2 + \frac{15}{2}x_3 \\ \text{subject to} & -1 \le x_j \le 1, \quad j = 1, 2, 3. \end{array}$$

An optimal extreme point to the subproblem is given by $v_3 = (-1 \;\; 1 \;\; -1)^T$ with optimal value -16. Hence, α_3 should enter the basis. The corresponding column in the master problem is given by

$$\begin{pmatrix} Av_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

The change to the basic variables is given by

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -\begin{pmatrix} 6 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Finding the maximum step η for which $\alpha + \eta p \ge 0$ gives

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \eta \begin{pmatrix} -2 \\ 1 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $\eta = 1/4$ so that α_1 leaves the basis.

Hence, the new basis corresponds to v_2 and v_3 so that

$$B = \begin{pmatrix} Av_3 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (0 \; 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 6 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_3 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_3 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_3 \; c^T v_2) = (-7 \; -3)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 6 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -4 \end{pmatrix}.$$

By forming $c^T - y_1 A = (5/2 \;\; 0 \;\; 3/2)$ we obtain the subproblem

$$\begin{array}{lll} 4+ & \text{minimize} & \frac{5}{2}x_1 + \frac{3}{2}x_3 \\ & \text{subject to} & -1 \le x_j \le 1, \quad j = 1,2,3. \end{array}$$

Both v_2 and v_3 are optimal extreme points to the subproblem, so the optimal value of the subproblem is 0. Hence, the master problem has been solved. The solution to the original problem is given by

$$v_3\alpha_3 + v_2\alpha_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \frac{1}{4} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \frac{3}{4} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ -1 \end{pmatrix}.$$

The optimal value is -4.

5. (a) We obtain

$$\begin{array}{rcll} \varphi(u) & = & \text{minimize} & (2-u)x_1 - 2(1+2u)x_2 + 3(1+u)x_3 \\ & & \text{subject to} & x_j \in \{-1, 0, 1\}, \quad j = 1,2,3. \end{array}$$

It is optimal to let x_j be plus or minus one, with opposite sign to the corresponding coefficient in the objective function. This gives

$$\varphi(u) = -|2-u| - 2|1+2u| - 3|1+u|.$$

The absolute value functions change sign at three distinct points, $u = -1$, $u = -1/2$ and $u = 2$.
If $u \le -1$, then $\varphi(u) = (u-2) + 2(1+2u) + 3(1+u) = 8u + 3$.
If $-1 \le u \le -1/2$, then $\varphi(u) = (u-2) + 2(1+2u) - 3(1+u) = 2u - 3$.
If $-1/2 \le u \le 2$, then $\varphi(u) = (u-2) - 2(1+2u) - 3(1+u) = -6u - 7$.
If $u \ge 2$, then $\varphi(u) = -(u-2) - 2(1+2u) - 3(1+u) = -8u - 3$.
Consequently, we obtain

$$\varphi(u) = \begin{cases} 8u+3 & \text{if } u \le -1, \\ 2u-3 & \text{if } -1 \le u \le -\frac{1}{2}, \\ -6u-7 & \text{if } -\frac{1}{2} \le u \le 2, \\ -8u-3 & \text{if } u \ge 2. \end{cases}$$

It follows that $u^* = -1/2$ is optimal to (D) with $\varphi(u^*) = -4$.

(b) For $u^* = -1/2$, the Lagrangian relaxation problem is given by

$$\begin{array}{rcll} \varphi(u^*) & = & \text{minimize} & \frac{5}{2}x_1 + \frac{3}{2}x_3 \\ & & \text{subject to} & x_j \in \{-1, 0, 1\}, \quad j = 1, 2, 3. \end{array}$$

Hence, it follows that $x_1(u^*) = -1$ and $x_3(u^*) = -1$ in an optimal solution, but $x_2(u^*)$ could be -1, 0 or 1. Hence, we get three optimal solutions, $x^1(u^*) = (-1 \ \ -1 \ \ -1)^T$, $x^2(u^*) = (-1 \ \ 0 \ \ -1)^T$ and $x^3(u^*) = (-1 \ \ 1 \ \ -1)^T$. By evaluating the relaxed constraint with reversed sign, $-x_1 - 4x_2 + 3x_3$, at these three points we get three subgradients, $s^1 = 2$, $s^2 = -2$ and $s^3 = -6$.

(c) Since the only constraint in the Lagrangian relaxation problem is $x_j \in \{-1, 0, 1\}$, $j = 1, 2, 3$, the optimal value will be unchanged if the integrality requirement is relaxed in that problem. Hence, the optimal values of the Lagrangian dual problem (D) and the LP relaxation (LP) will be identical. This is indeed the case, they are both -4.