SF2812 Applied linear optimization, final exam
Monday March 17 2014 8.00–13.00
Brief solutions

1. The basis corresponding to $\widetilde{y}$ and $\widetilde{s}$ is $\mathcal{B} = \{2, 3\}$. If b_1 is changed, the basis remains dual feasible. Hence, it is suitable to use the dual simplex method starting with this dual basic feasible solution. Let $y = \widetilde{y}$ and $s = \widetilde{s}$.

The basic variables are given by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives $x_2 = -1$, $x_3 = 1$. As $x_1 < 0$, the dual solution is not optimal. Consequently, since $x_2 < 0$, x_2 becomes nonbasic, and as x_1 is the first basic variable, the step in the y-direction is given by

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which gives $q_1 = -1$, $q_2 = 2$. With $y \leftarrow y + \alpha q$, dual feasibility requires $s \leftarrow s + \alpha\eta$, with $A^T q + \eta = 0$ and $s + \alpha\eta \geq 0$. Consequently, the nonnegativity of s requires $s - \alpha A^T q \geq 0$, i.e.,

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The maximum value of α is given by $\alpha_{\max} = 3$ making component 1 of $s - \alpha A^T q$ zero, so that the new basis becomes $\mathcal{B} = \{1, 3\}$. The basic variables are given by

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives $x_1 = 1$, $x_3 = 0$. As $x \geq 0$, an optimal solution has been obtained. Together with $y + \alpha_{\max} q$ and $s - \alpha_{\max} A^T q$ the primal and dual optimal solutions are given by

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}.$$

2. (See the course material.)

3. (a) The suggested initial extreme points $v_1 = (1 \;\; 0 \;\; 0)^T$ and $v_2 = (0 \;\; 0 \;\; 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^Tv_1\ c^Tv_2) = (-1\ 1)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

By forming $c^T - y_1 A = (-2\ \ -1\ \ -2)$ we obtain the subproblem

$$\begin{array}{lll} 2+ & \text{minimize} & -2x_1 - x_2 - 2x_3 \\ & \text{subject to} & x \in S. \end{array}$$

Both v_1 and v_2 are optimal extreme points to the subproblem, so that an optimal solution to the master problem has been found. The solution to the original problem is given by

$$v_1\alpha_1 + v_2\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{1}{2} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

The optimal value is 0.

(b) Given c_2, the subproblem is given by

$$\begin{array}{lll} 2+ & \text{minimize} & -2x_1 + (c_2 - 2)x_2 - 2x_3 \\ & \text{subject to} & x \in S. \end{array}$$

Hence, the subproblem has been solved as long as $c_2 - 2 \geq -2$, i.e., as long as $c_2 \geq 0$. For $c_2 < 0$, a new extreme point would enter the basis, $v_3 = (0\ \ 1\ \ 0)^T$.

4. (a) We have

$$\begin{array}{rll} \varphi(u) = u - & \text{maximize} & (2+u)x_1 + (3+u)x_2 + (3+u)x_3 \\ & \text{subject to} & x_1 + 2x_2 + 3x_3 \leq 2, \\ & & x_j \geq 0,\ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{array}$$

For this small problem, we may enumerate the feasible solutions. They are $(0\ \ 0\ \ 0)^T$, $(1\ \ 0\ \ 0)^T$, $(2\ \ 0\ \ 0)^T$, and $(0\ \ 1\ \ 0)^T$. Hence,

$$\varphi(u) = u - \max\{0, 2+u, 4+2u, 3+u\}.$$

Consequently, $\varphi(u) = u$ for $u \leq -3$, $\varphi(u) = -3$ for $-3 \leq u \leq -1$ and $\varphi(u) = -4 - u$ for $u \geq -1$. The corresponding optimal solutions to the problem that defines $\varphi(u)$ are $x(u) = (0\ 0\ 0)^T$ for $u \leq -3$, $x(u) = (0\ 1\ 0)^T$ for $-3 \leq u \leq -1$ and $x(u) = (2\ 0\ 0)^T$ for $u \geq -1$. (The optimal solution is nonunique for $u = -3$ and $u = -1$.)

(b) The dual problem is defined as

$$(D) \quad \begin{array}{ll} \underset{u \in I\!R}{\text{maximize}} & \varphi(u) \\ \text{subject to} & u \geq 0. \end{array}$$

Consequently, it is only $u \geq 0$ that is considered, and for these values of u, we have a relaxation. We do not consider $u < 0$.

(c) Since $\varphi(u) = -4 - u$ for $u \geq -1$, the dual problem takes the form

$$(D) \quad \begin{array}{ll} \underset{u \in I\!R}{\text{maximize}} & -4 - u \\ \text{subject to} & u \geq 0. \end{array}$$

The optimal solution is given by $u^* = 0$ with $\varphi(u^*) = -4$. By inspection, it has been found that $x = (2\ 0\ 0)^T$ is optimal to (IP) so that $\text{optval}(IP) = -4$. Hence, the duality gap is zero.

5. (a) Insertion of numerical values shows that the given x, y and s satisfy $Ax = b$, $x \geq 0$, $A^Ty + s = c$, $s \geq 0$, and $x_j s_j = 0$, $j = 1, 2, 3$. Hence, the optimality conditions are satisfied so x is optimal to (PLP) and (y, s) are optimal to (DLP).

(b) In order to identify an optimal extreme point, we may find a feasible variation around the current point, keeping the same constraints active. This means finding a direction p such that

$$\begin{pmatrix} 2 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Such a p is uniquely defined up to a scalar from the vector given in the hint, so we may let $p = (1\ 1\ 4\ 0)^T$. Since x is optimal and p is a feasible direction from x, it holds that $c^Tp = 0$. We may now identify optimal points with additional constraints active by considering $x + \alpha p$ for α positive and negative, i.e.,

$$\begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix}.$$

The most limiting negative value of α is $\alpha = -1$, for which we get the point $\bar{x} = (1\ 1\ 0\ 0)^T$. This point is an extreme point, since $(A_1\ A_2)$ has full column rank. However, since $p \geq 0$, there is no limit on α for $\alpha \geq 0$. In addition, starting from $\bar{x}$, the only constraint that may be deleted from the active constraints while maintaining optimality is $x_3 = 0$. Therefore, there is only one optimal extreme point, namely $\bar{x}$.

(c) By letting $\bar{\alpha} = \alpha - 1$ in the previous analysis, it follows that any optimal solution to (PLP) takes the form $\bar{x} + \bar{\alpha}p$ for $\bar{\alpha} \geq 0$. Optimality follows since $(\bar{x} + \bar{\alpha}p) = c^T\bar{x}$ independently of $\bar{\alpha}$.

Now consider a perturbed problem, where c_j is replaced by $c_j + \epsilon_j$, where ϵ_j is a "small positive number". The point is that since $c^Tp = 0$ and $0 \neq p \geq 0$, it follows that p becomes an ascent direction for this perturbed problem, i.e., $\sum_{j=1}^4 (c_j + \epsilon_j)p_j = \sum_{j=1}^4 \epsilon_j p_j > 0$, so that it is now optimal to let $\bar{\alpha} = 0$, making $\bar{x}$ the unique optimal solution. The technical details follow below, but these details are not expected from a student in the course.

The objective function value at $\bar{x} + \bar{\alpha} p$ for this perturbed problem is given by

$$\sum_{j=1}^{4}(c_j + \epsilon_j)(\bar{x}_j + \bar{\alpha} p_j) = \sum_{j=1}^{4}(c_j + \epsilon_j)\bar{x}_j + \bar{\alpha} \sum_{j=1}^{4}(c_j + \epsilon_j)p_j.$$

Taking into account $0 = c^T p = \sum_{j=1}^{4} c_j p_j$, it follows that

$$\sum_{j=1}^{4}(c_j + \epsilon_j)(\bar{x}_j + \bar{\alpha} p_j) = \sum_{j=1}^{4}(c_j + \epsilon_j)\bar{x}_j + \bar{\alpha} \sum_{j=1}^{4} \epsilon_j p_j.$$

But $\epsilon_j > 0$, $j = 1, 2, 3, 4$ and $0 \neq p \geq 0$ implies $\sum_{j=1}^{4} \epsilon_j p_j > 0$, so that

$$\sum_{j=1}^{4}(c_j + \epsilon_j)(\bar{x}_j + \bar{\alpha} p_j) > \sum_{j=1}^{4}(c_j + \epsilon_j)\bar{x}_j,$$

for $\bar{\alpha} > 0$. Therefore, deleting constraint $x_3 = 0$ at $\bar{x}$ results in a strict increase of objective function value for the perturbed problem. Hence, $\bar{x}$ is the unique optimal solution. The perturbation has to be sufficiently small so that s_4 remains positive for the perturbed problem.