SF2812 Applied linear optimization, final exam
Thursday May 22 2014 14.00–19.00
Brief solutions

1. (See the course material.)

2. (a) The system of primal-dual nonlinear equations is given by

$$
\begin{aligned}
x_1 + x_2 &= 1, &\text{(1a)}\\
y + s_1 &= 1, &\text{(1b)}\\
y + s_2 &= 3, &\text{(1c)}\\
x_1 s_1 &= \mu, &\text{(1d)}\\
x_2 s_2 &= \mu. &\text{(1e)}
\end{aligned}
$$

where we also implicitly require $x > 0$ and $s > 0$. We may use (1b)–(1e) to express x_1, x_2, s_1 and s_2 as a function of y according to

$$s_1 = 1 - y, \quad s_2 = 3 - y, \quad x_1 = \frac{\mu}{1-y}, \quad x_2 = \frac{\mu}{3-y}.$$

Insertion into (1a) gives $\frac{\mu}{1-y} + \frac{\mu}{3-y} = 1$ or equivalently

$$y^2 - 2(2-\mu)y + 3 - 4\mu = 0.$$

Solving this equation gives

$$y = 2 - \mu - \sqrt{(2-\mu)^2 - 3 + 4\mu} = y = 2 - \mu - \sqrt{1+\mu^2},$$

where the minus sign has been chosen to make $y < 1$, required by $s = 1 - y > 0$. With that we can, after simplification, express the solution as

$$
\begin{aligned}
x(\mu) &= \frac{1}{2}\begin{pmatrix} 1 - \mu + \sqrt{1+\mu^2} \\ 1 + \mu - \sqrt{1+\mu^2} \end{pmatrix},\\
y(\mu) &= 2 - \mu - \sqrt{1+\mu^2},\\
s(\mu) &= \begin{pmatrix} -1 + \mu + \sqrt{1+\mu^2}, \\ 1 + \mu + \sqrt{1+\mu^2} \end{pmatrix}.
\end{aligned}
$$

(b) Letting $\mu \to 0$ gives

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = 1, \quad s = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

It is straightforward to very that $Ax = b$, $x \geq 0$, $A^T y + s = c$, $s \geq 0$. Consequently, optimality holds.

3. The values of b_1 and b_2 must be such that $A\widehat{x} = b$, which gives $b_1 = 6$ and $b_2 = 10$. For these values of b_1 and b_2, the given $\widehat{x}$ is feasible.

The given $\widehat{x}$ is not a basic feasible solution. In order for $\widehat{x}$ to be optimal, there cannot be a basic feasible solution with lower objective function value. To find a basic feasible solution, we may compute directions in the null space of A_+, and successively add constraints. The v given in the hint is such that $A_+ v_+ = 0$, $v_0 = 0$.

Hence, if $\widehat{x}$ is optimal, it must hold that $c^T v = 0$. This implies that $c_1 = 3$. If we compute the maximum value of α such that of $\widehat{x} + \alpha v \geq 0$, we obtain $\alpha_{\max} = 1$. The point $\widehat{x} + \alpha_{\max} v$ has one more active constraint, and is in fact a basic feasible solution, with $x_1 = 4$ and $x_3 = 2$ as basic variables. The simplex multipliers are given by $B^T y = c_B$, i.e.,

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

which gives $y = (5 \;\; -2)^T$. The reduced costs are now given by $s = c - A^T y = (0\ 0\ 0\ c_4 + 3)^T$. Consequently, $s \geq 0$ if $c_4 \geq -3$. As the basic variables are strictly positive, it follows that the basic feasible solution is not optimal if $c_4 < -3$. Hence, we conclude that $\widehat{x}$ is optimal if and only if $b_1 = 6$, $b_2 = 10$, $c_1 = 3$ and $c_4 \geq -3$.

4. (a) For a fix vector $u \in I\!R^n$, Lagrangian relaxation of the first set of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n \left(-u_i + \sum_{j=1}^n (u_i - c_{ij}) x_{ij} \right) + \sum_{j=1}^n f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \leq b_j z_j, \quad j = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \ j = 1, \dots, n, \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

where a_i, $i = 1, \dots, n$, b_j, $j = 1, \dots, n$, f_j, $j = 1, \dots, n$, and c_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$, are nonnegative integer constants.

(b) For a fix nonnegative vector $v \in I\!R^n$, Lagrangian relaxation of the second group of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n \sum_{j=1}^n (a_i v_j - c_{ij}) x_{ij} + \sum_{j=1}^n (f_j - b_j v_j) z_j \\ \text{subject to} & \displaystyle\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \ j = 1, \dots, n, \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

where a_i, $i = 1, \dots, n$, b_j, $j = 1, \dots, n$, f_j, $j = 1, \dots, n$, and c_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$, are nonnegative integer constants.

(c) The first relaxation decomposes into one separate problem for each j according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n (u_i - c_{ij}) x_{ij} + f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \leq b_j z_j, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \\ & z_j \in \{0, 1\}, \end{array}$$

for $j = 1, \dots, n$. We can here solve two problems, for $z_j = 0$ and $z_j = 1$, and then take the minimum. For $z_j = 0$, the solution is given by $x_{ij} = 0$,

$j = 1, \dots, n$. For $z_j = 1$, we obtain a binary knapsack problem, which may for example be solved using dynamical programming.

The second relaxation decomposes into trivial problems. For the z-variables we obtain for each i according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{n} (f_j - b_j v_j) z_j \\ \text{subject to} & z_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

which can be solved directly with $z_j = 1$ if $f_j - b_j v_j < 0$ and $z_j = 0$ if $f_j - b_j v_j \geq 0$ for $j = 1, \dots, n$. For the x-variables we obtain

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{n} (a_i v_j - c_{ij}) x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1, \\ & x_{ij} \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

for $i = 1, \dots, n$. These can be solved directly by noting which x_{ij}-variable having the smallest coefficient in the objective function.

(d) The second relaxation gives a relaxed problem which gives integer optimal solutions even if one relaxes the integer constraint. Hence, the corresponding dual underestimation becomes identical with the one obtained if performing an LP-relaxation.

The first relaxation gives a more complicated relaxed problem, and here one can expect the underestimation to be better than one would obtain with an LP-relaxation.

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1} b = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T} e = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enters the basis.

The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{6}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 6\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 11, \\ & & \alpha_i \geq 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (2\ 1\ 0)^T$ with optimal value $-1/6$. Hence, $a_4 = (2\ 1\ 0)^T$ and the maximum step is given by

$$0 \leq x = B^{-1} b - \eta B^{-1} a_4 = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence, $\eta_{\max} = 30$ and x_1 leaves the basis, so that the basic variables are given by $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$. The reduced costs are given by

$$y = B^{-T}e = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 1$.

The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 11, \\ & & \alpha_i \ge 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$, with $a_2 = (0\ 2\ 0)^T$, $a_3 = (0\ 0\ 1)^T$ and $a_4 = (2\ 1\ 0)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 80 W-rolls, with 10 rolls cut according to pattern $(0\ 2\ 0)^T$, 40 rolls cut according to pattern $(0\ 0\ 1)^T$ and 30 rolls cut according to pattern $(2\ 1\ 0)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)