SF2812 Applied linear optimization, final exam
Wednesday March 18 2015 8.00–13.00
Brief solutions

1. (a) Insertion of numerical values gives $A\widehat{x} = b$. Hence, since $\widehat{x}$ is nonnegative, it is feasible.

Since v belongs to the nullspace of A, $\widehat{x} + \alpha v$ is feasible for all α such that $\widehat{x} + \alpha v$ is nonnegative. We have

$$x + \alpha v = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

By taking the limiting values of α for which $x + \alpha v$ remains nonnegative, we obtain $x^{(1)} = (1\ 3\ 2\ 0\ 0)^T$ for $\alpha^{(1)} = -1$ and $x^{(2)} = (3\ 1\ 0\ 2\ 0)^T$ for $\alpha^{(2)} = 1$. These solutions both have three positive components. In addition, the corresponding basis matrices are upper triangular, hence nonsingular. Therefore, both $x^{(1)}$ and $x^{(2)}$ are basic feasible solutions.

(b) We may use the basis provided by $x^{(1)}$ to calculate

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

i.e., $y = (1\ \ -1\ \ 0)^T$. Then, $s = c - A^T y = (0\ 0\ 0\ 0\ 3)^T$. Since $s \geq 0$, it follows that $x^{(1)}$ together with y and s form a primal and dual optimal pair. In addition, since $\widehat{x}$ and $x^{(1)}$ have the same objective function value, it follows that $\widehat{x}$ is optimal.

(c) Since $\widehat{x}_i > 0$ for $i = 1, 2, 3, 4$ and there must hold complementarity $\widehat{x}_j \widehat{s}_j = 0$, $j = 1, \dots, 5$, for any pair of primal and dual optimal solutions $\widehat{x}$ and $\widehat{y}$, $\widehat{s}$ respectively, we conclude that $\widehat{s}_j = 0$, $j = 1, 2, 3, 4$.

2. The suggested initial extreme points $v_1 = (1\ \ 0\ \ 0\ \ 0)^T$ and $v_2 = (0\ \ -1\ \ 0\ \ 0)^T$ give the initial basis matrix

$$B = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (-4\ \ 0)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

By forming $c^T - y_1 A = (-1 \;\; 1 \;\; 0 \; 1)$ we obtain the subproblem

$$\begin{array}{lll} 1+ & \text{minimize} & -x_1 + x_2 + x_4 \\ & \text{subject to} & \|x\|_1 = 1. \end{array}$$

The given extreme points v_1 and v_2 are optimal to the subproblem, so the optimal value of the subproblem is zero, and the master problem has been solved. The solution to the original problem is given by

$$v_1\alpha_1 + v_3\alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{3}{4} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{4} = \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ 0 \\ 0 \end{pmatrix}.$$

3. (a) With $X = \text{diag}(x)$ and $S = \text{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -14 \\ -1 \\ -3 \\ 0 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(b) If we compute $\alpha_{\max}$ as the largest step α for which $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s \geq 0$, the most limiting step is for component 3 in x, where $2 - 3.6\alpha \geq 0$ gives $\alpha_{\max} = 5/9$. As $\alpha_{\max} < 1$ we cannot accept the unit step. We may set

$\alpha = 0.99\alpha_{\max}$ for this value of $\alpha_{\max}$ which gives $\alpha = 0.55$. Then,

$$x^{(1)} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix} + 0.55 \begin{pmatrix} -2.4 \\ 0.4 \\ -3.6 \\ -0.4 \end{pmatrix} = \begin{pmatrix} 2.68 \\ 2.22 \\ 0.02 \\ 0.78 \end{pmatrix},$$
$$y^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.55 \begin{pmatrix} -0.6 \\ -1.0 \end{pmatrix} = \begin{pmatrix} -0.33 \\ -0.55 \end{pmatrix},$$
$$s^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix} + 0.55 \begin{pmatrix} 0.6 \\ -0.4 \\ 3.6 \\ 1.6 \end{pmatrix} = \begin{pmatrix} 1.33 \\ 1.78 \\ 3.98 \\ 4.88 \end{pmatrix}.$$

(The numerical values of $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ are not required.)

4. (See the course material.)

5. The basis corresponding to $\widetilde{y}$ and $\widetilde{s}$ is $\mathcal{B} = \{1, 2\}$. If b_1 is changed, the basis remains dual feasible. Hence, it is suitable to use the dual simplex method starting with this dual basic feasible solution. Let $y = \widetilde{y}$ and $s = \widetilde{s}$.

The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which gives $x_1 = -1$, $x_2 = 2$. As $x_1 < 0$, the dual solution is not optimal. Consequently, since $x_2 < 0$, x_2 becomes nonbasic, and as x_1 is the first basic variable, the step in the y-direction is given by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which gives $q_1 = -1$, $q_2 = 1$. With $y \leftarrow y + \alpha q$, dual feasibility requires $s \leftarrow s + \alpha \eta$, with $A^T q + \eta = 0$ and $s + \alpha \eta \geq 0$. Consequently, the nonnegativity of s requires $s - \alpha A^T q \geq 0$, i.e.,

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix} - \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The maximum value of α is given by $\alpha_{\max} = 1$ making component 4 of $s - \alpha A^T q$ zero, so that the new basis becomes $\mathcal{B} = \{2, 4\}$. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which gives $x_2 = 1/2$, $x_4 = 1/2$. As $x \geq 0$, an optimal solution has been obtained. Together with $y + \alpha_{\max} q$ and $s - \alpha_{\max} A^T q$ the primal and dual optimal solutions are given by

$$x = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$