SF2812 Applied linear optimization, final exam
Monday March 13 2017 8.00–13.00
Brief solutions

1. (a) From the GAMS output file, the values of "`VAR x`" suggest $x = (2\ 1\ 0\ 0\ 1)^T$, the marginal costs for "`EQU cons`" suggest $y = (2\ 1\ \ -1)^T$, and the marginal costs for "`VAR x`" suggest $s = (0\ 0\ 4\ 4\ 0)^T$. Insertion of numerical values gives $Ax = b$, $A^Ty + s = c$, $x \geq 0$, $s \geq 0$ and $x^Ts = 0$. Hence, the solutions are optimal to the respective problem.

(b) Both x_3 and x_4 are nonbasic variables. Consequently, a change of c_3 from 2 to $2 + \delta_3$ and a change of c_4 from 3 to $3 + \delta_4$ gives, for the same basis, $s_3 = 4 + \delta_3$ and $s_4 = 4 + \delta_4$, with other components of s, x and y unchanged. Consequently, it follows that the optimal solution is unchanged as long as the costs of x_3 or x_4 are not decreased more than 4 units. Hence, the solution is not at all sensitive to changes considered by AF. The computed optimal solution is optimal also considering the fluctuations. Therefore, there is no need for a stochastic programming model.

(c) Since $y_1 = 2$, the optimal value is expected to change with 2 per unit change of b_1.

2. (a) Insertion of numercial values gives $A\widehat{x} = b$, $A^T\widehat{y} + s = c$. In addition, $\widehat{x} \geq 0$, $\widehat{s} \geq 0$ and $\widehat{x}_j\widehat{s}_j = 0$, $j = 1, \ldots, 5$. Hence, the solutions are optimal to the primal and dual problems, respectively.

(b) The solution given by $\widehat{x}$ corresponds to x_1 and x_2 being basic variables. Since $\widehat{s}_3 = 0$, it follows that x_3 may enter the basis without changing the value of the objective function. Consequently, optimality is preserved. The corresponding direction is given by $p_3 = 1$ and

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which gives $p_1 = -1$ and $p_2 = 1$. By setting $x_B + \alpha p_B \geq 0$, we obtain $\alpha_{\max} = 1$. Consequently, new basic variables are $x_2 = 3$ and $x_3 = 2$.
We may compute y and s from $B^Ty = c_B$, $s = c - A^Ty$, i.e.,

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

with solution $y = (2\ \ -1)^T$, so that $s = c - A^Ty = (0\ \ 0\ \ 0\ \ 3)^T$. As $s \geq 0$, we have an optimal solution. In addition, since $s_1 = 0$ but $s_4 > 0$, it follows that it is only x_1 that may enter the basis again without increasing the objective function value. This would give us $\widehat{x}$ back. Consequently, there are only two optimal basic feasible solutions, $\widehat{x}$ and $(0\ \ 2\ \ 1\ \ 0)^T$. Therefore, the set of optimal solutions is given by the set of convex combinations of these points, i.e.,

$$\left\{ (1-\alpha) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} : 0 \leq \alpha \leq 1 \right\}.$$

By comparing to the given $x(\mu)$, it follows that $x(\mu)$ is close to the optimal solution given by $\alpha = 0.5773$.

As the barrier trajectory avoids active constraints, $x(\mu)$ will converge to a basic feasible solution when $\mu \to 0$ only if the optimal solution is unique. This is not the case here.

3. (See the course material.)

4. (a) The dual objective $\varphi(v)$ is the optimal solution of

$$\begin{aligned}
& \text{minimize } -x_1 - 4x_3 - x_4 + v_1(x_1 + x_2 - 1) + v_2(x_3 + x_4 - 1) \\
& \text{subject to } 4x_1 + 7x_2 + 6x_3 + 5x_4 \le 10,\ x_j \in \{0,1\},\ j = 1,\dots,4, \\
= \ & -v_1 - v_2 - \text{maximize } (1 - v_1)x_1 - v_1x_2 + (4 - v_2)x_3 + (1 - v_2)x_4 \\
& \qquad \text{subject to } 4x_1 + 7x_2 + 6x_3 + 5x_4 \le 10, \\
& \qquad x_j \in \{0,1\},\ j = 1,\dots,4.
\end{aligned}$$

In particular, for $v = \widehat{v}$, we obtain

$$\begin{aligned}
\varphi(\widehat{v}) = -3 - \text{maximize } & -x_2 + 2x_3 - x_4 \\
\text{subject to } & 4x_1 + 7x_2 + 6x_3 + 5x_4 \le 10, \\
& x_j \in \{0,1\},\ j = 1,\dots,4.
\end{aligned}$$

It follows that $x_2 = 0$ and $x_4 = 0$ in all optimal solutions, since the corresponding objective function coefficients are negative in the maximization problem. Hence, we obtain two optimal solutions, $x^{(1)}(\widehat{v}) = (0\ 0\ 1\ 0)^T$ and $x^{(2)}(\widehat{v}) = (1\ 0\ 1\ 0)^T$ with $\varphi(\widehat{v}) = -5$.

(b) We obtain two subgradients $s^{(1)}$ and $s^{(2)}$ to φ at $\widehat{v}$ by evaluating the relaxed constraints with reversed sign at $x^{(1)}(\widehat{v})$ and $x^{(2)}(\widehat{v})$ respectively, as

$$\begin{aligned}
s^{(1)} &= -\begin{pmatrix} 1 - x_1^{(1)}(\widehat{v}) - x_2^{(1)}(\widehat{v}) \\ 1 - x_3^{(1)}(\widehat{v}) - x_4^{(1)}(\widehat{v}) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\
s^{(2)} &= -\begin{pmatrix} 1 - x_1^{(2)}(\widehat{v}) - x_2^{(2)}(\widehat{v}) \\ 1 - x_3^{(2)}(\widehat{v}) - x_4^{(2)}(\widehat{v}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

(c) As $s^{(2)} = 0$, it follows that $\widehat{v}$ is optimal to the dual problem.

5. (a) The maximization inside the constraint, $\max_{v_i \in \mathcal{P}_i}\{v_i^T y\}$, is a linear program on the form

$$\begin{aligned}
\underset{v_i \in I\!R^m}{\text{maximize}} \quad & y^T v_i \\
\text{subject to} \quad & C_i^T v_i \le d_i,
\end{aligned}$$

where y is fixed and v_i is the variable vector. The corresponding dual problem takes the form

$$\begin{array}{ll} \underset{z_i \in I\!R^{n_i}}{\text{minimize}} & d_i^T z_i \\ \text{subject to} & C_i z_i = y, \\ & z_i \geq 0. \end{array}$$

By strong duality for linear programming, the optimal values of these two linear programs are identical. Consequently, the requirement $\max_{v_i \in \mathcal{P}_i} \{v_i^T y\} \leq c_i$ is equivalent to the existence of a $z_i \in I\!R^{n_i}$ such that

$$\begin{aligned} d_i^T z_i &\leq c_i, \\ C_i z_i &= y, \\ z_i &\geq 0. \end{aligned}$$

We may therefore equivalently formulate (RP) as a linear program on the form

$$(LPR) \quad \begin{array}{lll} \text{maximize} & b^T y & \\ \text{subject to} & d_i^T z_i \leq c_i, & i = 1, \dots, n, \\ & C_i z_i - y = 0, & i = 1, \dots, n, \\ & z_i \geq 0, & i = 1, \dots, n. \end{array}$$

In order to derive the dual problem associated with (LPR), we may introduce nonnegative Lagrange multipliers $\alpha_i \in I\!R$, associated with the constraints $c_i - d_i^T z_i \geq 0$, and multipliers $\beta_i \in I\!R_i^n$, associated with the constraints $C_i z_i - y = 0$. Lagrangian relaxation then gives the dual objective function

$$\begin{aligned} & \max_{y, z_1 \geq 0, \dots, z_n \geq 0} \{b^T y + \sum_{i=1}^n \alpha_i (c_i - d_i^T z_i) + \sum_{i=1}^n \beta_i^T (C_i z_i - y)\} \\ = & \sum_{i=1}^n c_i \alpha_i + \max_y \{(b - \sum_{i=1}^n \beta_i)^T y\} + \sum_{i=1}^n \max_{z_i \geq 0} \{(C_i^T \beta_i - d_i \alpha_i)^T z_i\} \\ = & \begin{cases} \sum_{i=1}^n c_i \alpha_i & \text{if } C_i^T \beta_i - d_i \alpha_i \leq 0, \ i = 1, \dots, n, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem therefore takes the form

$$(DLPR) \quad \begin{array}{lll} \text{minimize} & \sum_{i=1}^n c_i \alpha_i & \\ \text{subject to} & \sum_{i=1}^n \beta_i = b, & \\ & C_i^T \beta_i - d_i \alpha_i \leq 0, & i = 1, \dots, n, \\ & \alpha_i \geq 0, & i = 1, \dots, n. \end{array}$$

(b) For this particular case, the constraint $C_i^T \beta_i - d_i \alpha_i \leq 0$ takes the form

$$\begin{pmatrix} I \\ -I \end{pmatrix} \beta_i - \begin{pmatrix} A_i \\ -A_i \end{pmatrix} \alpha_i \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to $\beta_i = A_i \alpha_i$. We may therefore eliminate β_i, $i = 1, \dots, n$, and write the dual problem as

$$(DLPR) \quad \begin{array}{lll} \text{minimize} & \sum_{i=1}^n c_i \alpha_i & \\ \text{subject to} & \sum_{i=1}^n A_i \alpha_i = b, & \\ & \alpha_i \geq 0, & i = 1, \dots, n, \end{array}$$

which is the dual problem associated with (LP). This is what we would expect, as in this case (RP) is equivalent to (LP).