SF2812 Applied linear optimization, final exam
Wednesday June 7 2017 14.00–19.00
Brief solutions

1. (a) A solution $\widetilde{x}$ and Lagrange multiplier vector $\widetilde{y}$ associated with (P_μ) also solve the primal-dual nonlinear equations. Therefore,

$$x(\mu) = \widetilde{x} \approx \begin{pmatrix} 1.9906 \\ 0.0010 \\ 0.9956 \\ 0.9993 \\ 0.0005 \\ 0.0003 \end{pmatrix}, \quad y(\mu) = \widetilde{y} \approx \begin{pmatrix} 1.9987 \\ -0.9993 \\ 0.9992 \end{pmatrix}.$$

Finally, we may obtain $s(\mu)$ from $s_j(\mu) = \mu / x_j(\mu)$, $j = 1, \ldots, 6$. From the hint, it follows that

$$s(\mu) \approx \begin{pmatrix} 0.0005 \\ 1.0064 \\ 0.0010 \\ 0.0010 \\ 2.0040 \\ 3.0045 \end{pmatrix}.$$

(b) We expect the solutions to be in the order of 10^{-3} away from an optimal solution. Therefore, rounding gives

$$x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 3 \end{pmatrix}.$$

We have $Ax = b$, $A^T y + s = c$, $x \geq 0$, $s \geq 0$ and $x^T s = 0$. Hence, the solutions are optimal to the respective problem.

(c) The computed solution is a basic feasible solution. In addition, since strict complementarity holds, the solution is unique. Consequently, the simplex method would compute the same solution.

2. (a) The simplex multipliers are given by

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

which gives $y_1 = 0$, $y_2 = 1$. The reduced costs are then given by

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This is a dual feasibls solution, because $s \geq 0$.

(b) The corresponding basic variables are given by

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$

which gives $x_2 = 1$, $x_3 = -3$.

As $x_3 < 0$, the dual solution is not optimal. If q denotes the step in the y-direction and η denotes the step in the s-direction, dual feasibility requires $A^T q + \eta = 0$. Consequently, since $x_3 < 0$, x_3 becomes nonbasic, and we obtain

$$\eta_B = \begin{pmatrix} \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

. The step in the y-direction is given by $B^T q = -\eta_B$, i.e.,

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

which gives $q_1 = 1$, $q_2 = -1/2$. With $y \leftarrow y + \alpha q$, dual feasibility requires $s \leftarrow s + \alpha\eta$, with $A^T q + \eta = 0$ and $s + \alpha\eta \geq 0$. Consequently, the nonnegativity of s requires $s - \alpha A^T q \geq 0$, i.e.,

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} \frac{3}{2} \\ 0 \\ -1 \\ \frac{1}{2} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The maximum value of α is given by $\alpha_{\max} = 4/3$ making component 1 of $s - \alpha A^T q$ zero, so that the new basis becomes $\mathcal{B} = \{1, 2\}$. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$

which gives $x_1 = 2$, $x_2 = 2$. As $x \geq 0$, an optimal solution has been obtained. Together with $y + \alpha_{\max} q$ and $s - \alpha_{\max} A^T q$ the primal and dual optimal solutions are given by

$$x = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 \\ 0 \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}.$$

3. (See the course material.)

4. (a) We have

$$\begin{aligned} \varphi(u) = u - \quad & \text{maximize} && (3+u)x_1 + (4+u)x_2 + (3+u)x_3 \\ & \text{subject to} && x_1 + 2x_2 + 3x_3 \leq 2, \\ & && x_j \geq 0, \ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{aligned}$$

For this small problem, we may enumerate the feasible solutions. They are $(0\;\;0\;\;0)^T$, $(1\;\;0\;\;0)^T$, $(2\;\;0\;\;0)^T$, and $(0\;\;1\;\;0)^T$. Hence,

$$\varphi(u) = u - \max\{0, 3+u, 6+2u, 4+u\}.$$

Consequently, $\varphi(u) = u$ for $u \le -4$, $\varphi(u) = -4$ for $-4 \le u \le -2$ and $\varphi(u) = -6 - 2u$ for $u \ge -2$. The corresponding optimal solutions to the problem that defines $\varphi(u)$ are $x(u) = (0\;\;0\;\;0)^T$ for $u \le -4$, $x(u) = (0\;\;1\;\;0)^T$ for $-4 \le u \le -2$ and $x(u) = (2\;\;0\;\;0)^T$ for $u \ge -2$. (The optimal solution is nonunique for $u = -4$ and $u = -2$.)

(b) The dual problem is defined as

$$(D) \quad \begin{array}{ll} \underset{u \in I\!R}{\text{maximize}} & \varphi(u) \\ \text{subject to} & u \ge 0. \end{array}$$

Consequently, it is only $u \ge 0$ that is considered, and for these values of u, we have a relaxation. We do not consider $u < 0$.

(c) Since $\varphi(u) = -6 - 2u$ for $u \ge -2$, the dual problem takes the form

$$(D) \quad \begin{array}{ll} \underset{u \in I\!R}{\text{maximize}} & -6 - 2u \\ \text{subject to} & u \ge 0. \end{array}$$

The optimal solution is given by $u^* = 0$ with $\varphi(u^*) = -6$. By inspection, it has been found that $x = (2\;0\;0)^T$ is optimal to (IP) so that $\text{optval}(IP) = -6$. Hence, the duality gap is zero.

5. (a) For the given cut patterns, we obtain

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\;1\;1)^T$. As $y \ge 0$ no slack variables enters the basis.
The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{6}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 6\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 11, \\ & & \alpha_i \ge 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (2\;1\;0)^T$ with optimal value $-1/6$. Hence, $a_4 = (2\;1\;0)^T$ and the maximum step is given by

$$0 \le x = B^{-1}b - \eta B^{-1}a_4 = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence, $\eta_{\max} = 30$ and x_1 leaves the basis, so that the basic variables are given by $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$. The reduced costs are given by

$$y = B^{-T}e = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 1$.
The subproblem is given by

$$1 - \tfrac{1}{4}\begin{array}{ll} \text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 11, \\ & \alpha_i \ge 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$, with $a_2 = (0\ 2\ 0)^T$, $a_3 = (0\ 0\ 1)^T$ and $a_4 = (2\ 1\ 0)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 80 W-rolls, with 10 rolls cut according to pattern $(0\ 2\ 0)^T$, 40 rolls cut according to pattern $(0\ 0\ 1)^T$ and 30 rolls cut according to pattern $(2\ 1\ 0)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)