SF2812 Applied linear optimization, final exam
Tuesday June 5 2018 14.00–19.00
Brief solutions

1. (a) Since $\widetilde{y}, \widetilde{s}$ is a feasible solution to (DLP) and (DLP) is a maximization problem, $b^T\widetilde{y}$ is a lower bound for optval(DLP).

(b) By strong duality for linear programming, the optimal values of (PLP) and (DLP) are equal, if both problems are feasible. Therefore, $b^T\widetilde{y}$ is a lower bound for optval(PLP). There is no implication that optval$(PLP) < \infty$ by existence of dual feasible solution.

(c) It holds that $\widetilde{y}+\alpha\eta$, $\widetilde{s}+\alpha q$ is feasible for all $\alpha \ge 0$, since $A^T(\widetilde{y}+\alpha\eta)+\widetilde{s}+\alpha q = c$ and $\widetilde{s}+\alpha q \ge 0$ for $\alpha \ge 0$. Since $b^T(y+\alpha\eta)$ tends to infinity as $\alpha \to \infty$, we conclude that optval$(PLP) =$ optval$(DLP) = \infty$.

(d) If $\widetilde{x}$ is feasible to (PLP) and $\widetilde{y}$, $\widetilde{s}$ is feasible to (DLP), it holds that $\widetilde{x}^T\widetilde{s} = c^T\widetilde{x} - b^T\widetilde{y}$. Therefore, by strong duality for linear programming, we must have $\widetilde{x}^T\widetilde{s} = 0$ if the solutions are optimal to the respective problems. Therefore, if $\widetilde{x}$ is optimal to (PLP) and $\widetilde{x}^T\widetilde{s} = 1$, it cannot hold that $\widetilde{y}$, $\widetilde{s}$ is optimal to (DLP).

2. (a) The primal variables x are given by the values ("`LEVEL`") of "`VAR x`" as $x = (0\ 2\ 1\ 1\ 3\ 0)^T$. The dual variables y are given as the the marginal values of the constraints $Ax = b$, i.e., the marginal values ("`MARGINAL`") of "`EQU cons`", $y = (1\ {-1}\ \ 1\ {-1})^T$. The dual variables s are given as the the marginal values of the constraints $x \ge 0$, i.e., the marginal values ("`MARGINAL`") of "`VAR x`", $s = (2\ 0\ 0\ 0\ 0\ 5))^T$. The GAMS output file gives "`MODEL STATUS Optimal`", so the solutions are optimal.

(b) We see that components 2, 3, 4, and 5 of x are positive. The corresponding columns of A form a triangular nonsingular basis matrix B. As long as the change in b gives the same optimal basis, strong duality shows that the change in optimal value is given by $b^Ty + \delta e_2^T y + \delta e_3^T y$, i.e., 4.

(c) We have $Bx_B = b$. If b is changed to $b + \delta e_2 + \delta e_3$, we get the corresponding primal solution x_B^δ by $Bx_B^\delta = b + \delta e_2 + \delta e_3$, i.e., $x_B^\delta = x_B + \delta p_B$, where $Bp_B = e_2 + e_3$. Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The solution is given by $p_B = (0\ 1\ 0\ {-1})^T$. The bound on δ is then given by primal feasibility, i.e., $x_B + \delta p_B \ge 0$. Insertion of numerical values gives

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

i.e.,

$$\begin{aligned} 1 + \delta &\ge 0, \\ 3 - \delta &\ge 0. \end{aligned}$$

The bound is consequently given by $-1 \le \delta \le 3$. Therefore, the optimal value is given by 4 for $-1 \le \delta \le 3$.

3. (a) The optimality conditions of (P_μ) may be written as

$$c - \mu X^{-1} e = A^T y,$$
$$Ax = b,$$

in addition to $x > 0$, where $X = \mathrm{diag}(x)$ and e is the vector of ones.
It is given that $\widetilde{x}$ is feasible, so $A\widetilde{x} = b$ holds.
The matrix Z is a 4×2 matrix of full column rank such that $AZ = 0$. Hence, since A is a 2×4 matrix of full row rank, the columns of Z form a basis for the nullspace of A. Therefore, the condition $c - \mu X^{-1} e = A^T y$ is equivalent to $Z^T(c - \mu X^{-1} e) = 0$.
Evaluation gives $Z^T c = (3\ 1)^T$, so that

$$Z^T(c - \mu \tilde{X}^{-1} e) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 0.1 \cdot \begin{pmatrix} 30 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

verifying the second optimality condition for $\mu = 0.1$. Finally, $\widetilde{x} > 0$. Therefore, $\widetilde{x}$ is optimal to (P_μ), i.e., $\widetilde{x} = x(\mu)$ for $\mu = 0.1$.

(b) In case of strict complementarity, we expect $x(\mu)$ to differ by $O(\mu)$ from the optimal solution x^*. Since $\widetilde{x} = x(\mu)$ for $\mu = 0.1$, we expect $O(\mu) \approx 0.1$, and therefore guess $x^* = (0\ 5\ 0\ 10)^T$. This would correspond to x_2 and x_4 being basic variables. Then, $A_B^T y^* = c_B$ gives

$$\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1^* \\ y_2^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e., $y^* = (1\ 0)^T$. Evaluating $s^* = c - A^T y^*$ gives $s^* = (1\ 0\ 1\ 0)^T$. Since $s^* \ge 0$, we conclude that x^* is optimal to (LP).

(c) The primal-dual system of nonlinear equations take the form

$$\begin{aligned} A^T y + s - c &= 0, \\ Ax - b &= 0, \\ XSe - \mu e &= 0. \end{aligned}$$

They are equivalent to the optimality conditions of (P_μ). Therefore, we know $x(\mu)$, since $x(\mu) = \widetilde{x}$. We therefore need to find $s(\mu)$ from

$$s_i(\mu) = \frac{\mu}{x_i(\mu)} = \frac{\mu}{\widetilde{x}_i},, \quad i = 1, \ldots, 4,$$

and $y(\mu)$ from the relation

$$c - s(\mu) = A^T y(\mu).$$

We get

$$s(\mu) \approx 0.1 \begin{pmatrix} 10.9135 \\ 0.2032 \\ 9.4925 \\ 0.1014 \end{pmatrix} \approx \begin{pmatrix} 1.0914 \\ 0.0203 \\ 0.9493 \\ 0.0101 \end{pmatrix}.$$

Finally,

$$c - s(\mu) \approx \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1.0914 \\ 0.0203 \\ 0.9493 \\ 0.0101 \end{pmatrix} \approx \begin{pmatrix} 1.9086 \\ 0.9797 \\ -0.9493 \\ -0.0101 \end{pmatrix},$$

so that

$$\begin{pmatrix} 1.9087 \\ 0.9797 \\ -0.9492 \\ -0.0101 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1(\mu) \\ y_2(\mu) \end{pmatrix}.$$

The last two equations give

$$y(\mu) \approx \begin{pmatrix} 0.9492 \\ 0.0101 \end{pmatrix}.$$

Since $y(\mu)$ is unique, and we know there is a solution, we need not verify the first two equations.
(As a check, we note that $y(\mu)$ is close to y^* and $s(\mu)$ is close to s^*, by $O(\mu) \approx 0.1$.)

4. (See the course material.)

5. (a) The given cut pattern give an initial basis in the master problem, corresponding to a basic feasible solution. We obtain

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 15 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enter the basis.
The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 12, \\ & & \alpha_i \geq 0, \text{ integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (1\ 0\ 1)^T$ with optimal value $-1/4$. Hence, $a_4 = (1\ 0\ 1)^T$ and the maximum step is given by

$$0 \leq x = B^{-1}b - \eta B^{-1}a_4 = \begin{pmatrix} 15 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{1}{4} \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $\eta_{\max} = 40$ and x_3 leaves the basis, so that the basic variables are given by $x_1 = 5$, $x_2 = 25$ and $x_4 = 40$. The simplex multipliers y are given by

$$y = B^{-T}e = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 3/4$.
The subproblem is given by

$$\begin{array}{lll} 1 - & \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ & \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \le 12, \\ & & \alpha_i \ge 0, \ \text{integer}, \quad i = 1, 2, 3. \end{array}$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_1 = 5$, $x_2 = 25$ and $x_4 = 40$, with $a_1 = (4\ 0\ 0)^T$, $a_2 = (0\ 2\ 0)^T$ and $a_4 = (1\ 0\ 1)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 70 W-rolls, with 5 W-rolls cut according to pattern $(4\ 0\ 0)^T$, 25 W-rolls cut according to pattern $(0\ 2\ 0)^T$ and 40 W-rolls cut according to pattern $(1\ 0\ 1)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)