KTH Mathematics

SF2812 Applied linear optimization, final exam
Friday January 15 2010 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear program (LP) defined as

$$(LP) \qquad \begin{array}{ll} \min & x_1 + 2x_2 \\ \text{då} & x_1 + 3x_2 = 1, \\ & x_1 \geq 0, \;\; x_2 \geq 0. \end{array}$$

Consider a barrier transformation of (LP) for a given positive barrier parameter μ.

(a) For the given μ, formulate the primal-dual system of nonlinear equations corresponding to the problem above. Use the fact that the problem is small to calculate an explicit expression for the solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to these nonlinear equations. .... (8p)

(b) Show that the calculated $x(\mu)$, $y(\mu)$ and $s(\mu)$ converge to an optimal solution of (LP) and corresponding dual problem as $\mu \to 0$. .... (2p)

2. Consider a binary knapsack problem (KP) defined as

$$(KP) \qquad \begin{array}{ll} \text{minimize} & -\displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{subject to} & -\displaystyle\sum_{j=1}^{n} a_j x_j \geq -b, \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n, \end{array}$$

where $a \geq 0$, $c \geq 0$ and $b \geq 0$.

(a) Give an explicit expression for the objective function $\varphi(\lambda)$ of the dual problem (D) arising when the constraint $-\sum_{j=1}^{n} a_j x_j \geq -b$ is relaxed by Lagrangian relaxation. ....(3p)

(b) For a given $\lambda \in I\!R$, find an explicit expression for a subgradient to φ at λ. (2p)

(c) Assume that $n = 3$, $a = (2\ 3\ 4)^T$, $b = 6$ and $c = (4\ 5\ 6)^T$. Illustrate the dual problem graphically. From the figure calculate the optimal solution and the optimal objective value of the dual problem. Solve the small (KP) by inspection, and determine the duality gap. ....(5p)

3. Consider the stochastic program (P) given by

$$(P)\qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \ldots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \ldots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) ....(6p)

(b) Define *VSS* in terms of suitable optimization problems. ....(2p)

(c) Define *EVPI* in terms of suitable optimization problems. ....(2p)

4. Consider a transportation problem (PTP) defined as

$$(PTP)\qquad \begin{array}{ll} \text{minimize} & \sum_{i=1}^{4}\sum_{j=1}^{3} c_{ij} x_{ij} \\ \text{subject to} & \sum_{j=1}^{3} x_{ij} = a_i, \quad i = 1,2,3,4, \\ & \sum_{i=1}^{4} x_{ij} = b_j, \quad j = 1,2,3, \\ & x_{ij} \geq 0, \quad i = 1,2,3,4,\ j = 1,2,3, \end{array}$$

with corresponding dual problem (DTP) given by

$$(DTP) \quad \begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{4} a_i u_i + \sum_{j=1}^{3} b_j v_j \\ \text{subject to} & u_i + v_j + s_{ij} = c_{ij}, \quad i = 1,2,3,4, \ j = 1,2,3, \\ & s_{ij} \geq 0, \quad i = 1,2,3,4, \ j = 1,2,3, \end{array}$$

where

$$C = \begin{pmatrix} 8 & 6 & 10 \\ 9 & 1 & 10 \\ 1 & 3 & 2 \\ 9 & 5 & 10 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}.$$

AF has implemented a primal-dual interior method for linear programming. When solving this transportation problem he obtains

$$\widehat{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 9 \\ \frac{15}{2} \\ 2 \\ 10 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} -1 \\ -\frac{13}{2} \\ 0 \end{pmatrix}, \quad \widehat{S} = \begin{pmatrix} 0 & \frac{7}{2} & 1 \\ \frac{5}{2} & 0 & \frac{5}{2} \\ 0 & \frac{15}{2} & 0 \\ 0 & \frac{3}{2} & 0 \end{pmatrix}.$$

(a) Show that $\widehat{X}$ is an optimal solution to (PTP). .... (3p)

(b) AF is a bit confused. He expected the optimal X to have integer values only, as he is solving a transportation problem with a and b integer valued. Explain why it is not surprising that AF's interior solver has not produced an optimal X with integer values only. .... (2p)

(c) Find, using $\widehat{X}$, two integer optimal solutions to (PTP). .... (3p)
Hint: The matrix P given by

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

might be helpful.

(d) If (PTP) was solved by the simplex method, which of the three optimal solutions mentioned above might be obtained? Motivate your answer. .... (2p)

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(3\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!