SF2812 Applied linear optimization, final exam
Monday March 17 2014 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear program

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix}^T.$$

An optimal basic feasible solution has been computed for $b_1 = 3$. This solution is $\widetilde{x} = (0\ 1\ 1)^T$. The corresponding dual optimal solution is $\widetilde{y} = (2\ \ -1)^T$ and $\widetilde{s} = (3\ 0\ 0)^T$.

Unfortunately, the value of b_1 was not correct. The correct value is $b_1 = 1$. Solve the correct problem by the dual simplex method, making use of the already computed solutions. Motivate why the dual form of the simplex method is suitable. . . . (10p)

2. Consider the linear program (LP) defined by

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where A is a given $m \times n$-matrix with linearly independent rows. Let $S = \{x : Ax = b, x \geq 0\}$.

(a) Define a convex set. . . . (2p)

(b) Show that S is a convex set. . . . (2p)

(c) Define a basic feasible solution to (LP). . . . (2p)

(d) Show that x is an extreme point to S if and only if x is a basic feasible solution to (LP). . . . (4p)

3. Consider the linear program (LP) given by

$$(LP)\quad \begin{array}{ll} \text{minimize} & -x_1 + x_2 + x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 = 2, \\ & x \in S, \end{array}$$

where $S = \{x \in I\!R^3 : x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.

(a) Solve (LP) using Dantzig-Wolfe decomposition taking into account problem structure.
Treat the equality constraint $x_1 + 2x_2 + 3x_3 = 2$ as the hard constraint. Write $x \in S$ as a convex combination of the extreme points of S. You may make use of the fact that S is bounded. In the master problem, start with the basis that corresponds to the extreme points $(1\ \ 0\ \ 0)^T$ and $(0\ 0\ 1)^T$. The subproblem(s) that arise(s) may be solved by inspection. .... (7p)

(b) The coefficient for x_2 in the objective function, c_2, is given by $c_2 = 1$. Based on the solution that you have given in (3a), determine all values of c_2 for which the solution that you computed remains optimal. .... (3p)

4. Consider the integer programming problem (IP) given by

$$(IP)\quad \begin{array}{ll} \text{minimize} & -2x_1 - 3x_2 - 3x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \geq 1, \\ & -x_1 - 2x_2 - 3x_3 \geq -2, \\ & x_j \geq 0,\ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{array}$$

For $u \in I\!R$, let

$$\varphi(u) = \begin{array}{ll} \text{minimize} & -2x_1 - 3x_2 - 3x_3 - u(x_1 + x_2 + x_3 - 1) \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \geq -2, \\ & x_j \geq 0,\ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{array}$$

You may throughout this exercise use the fact that the problem is small and your methods for solving subproblems that arise need not by systematic.

(a) Determine $\varphi(u)$ for $u \in I\!R$. .... (3p)

(b) Your friend AF is a bit confused. By inspection, he can see that an optimal solution to (IP) is given by $x = (2\ 0\ 0)^T$ so that $\text{optval}(IP) = -4$, where $\text{optval}(IP)$ denotes the optimal value of (IP). By his calculations, he has $\varphi(-1) = -3$. Explain to him why it is not a contradiction to our theory on Lagrangian relaxation that there exists a $u \in I\!R$ such that $\varphi(u) > \text{optval}(IP)$. .... (3p)

(c) Determine an optimal solution to the dual problem that results when the constraint $x_1 + x_2 + x_3 \geq 1$ is relaxed by Lagrangian relaxation. In addition, determine the duality gap. .... (4p)

5. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 3 & -1 & 2 \end{pmatrix}^T.$$

AF has implemented a primal-dual interior method in Matlab. He has tried to solve the above linear program by his solver and obtained the following approximate numbers for x, y, and s:

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x' =  2.0000    2.0000    4.0000    0.0000

y' =  1.0000   -1.0000

s' =  0.0000    0.0000    0.0000    1.0000
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Based on these results, AF claims that $x = (2\,2\,4\,0)^T$, $y = (1\,-1)^T$ and $s = (0\,0\,0\,1)^T$ give optimal solutions to (PLP) and (DLP) respectively.

AF is a bit confused by the results based on conversations with TO, who gave him the problem.

(a) Verify that AF has indeed found an optimal solution. .... (2p)

(b) TO has told AF that (PLP) has only one extreme point which is optimal. Since the solution computed by AF is not an extreme point, AF is confused. AF expects any optimal solution to be written as a convex combination of optimal extreme points. Hence, there should be at least two optimal extreme points. Who is right? Explain the situation to AF. .... (5p)
Hint: You may find the following result useful,

$$\begin{pmatrix} 2 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) TO also claims that AF's primal-dual interior method would give an approximation of an optimal extreme-point solution to (PLP) if AF added small positive numbers to the coefficients of c, where TO with a "small positive number" means a number which is significantly smaller than 1 but larger than his optimality tolerance. Is TO right? .... (3p)

Good luck!