SF2812 Applied linear optimization, final exam
Wednesday June 10 2015 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) be defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} -8 & 2 & 0 & 2 & -2 \end{pmatrix}^T.$$

(a) A person named AF has used GAMS to model and solve this problem. The GAMS input file can be found at the end of the exam. Unfortunately, AF has lost the GAMS output file. He does have a partial GAMS output file, which reads:

```
---- EQU cons  constraints

      LOWER     LEVEL     UPPER    MARGINAL

i1     4.000     4.000     4.000    -1.000
i2     1.000     1.000     1.000     2.000
i3     5.000     5.000     5.000    -2.000

                       LOWER     LEVEL     UPPER    MARGINAL

---- EQU objfun          .         .         .       -1.000

  objfun  objective function

                       LOWER     LEVEL     UPPER    MARGINAL

---- VAR objval        -INF    -12.000     +INF        .

  objval  objective function value

---- VAR x  decision variables
```

```
        LOWER     LEVEL     UPPER    MARGINAL

j1        .        2.000     +INF        .
j2        .          .       +INF      3.000
j3        .          .       +INF      1.000
j4        .        3.000     +INF        .
j5        .        1.000     +INF        .
```

AF has run several versions of the problem, and he is not sure that this file is the one that corresponds to (LP). Help AF by showing that this file gives the optimal solution to (LP). ....(4p)

(b) The reason that AF has run several versions of the file is that the coefficients for x_2 and x_3 in the objective function are not very precise. Hence, in addition to wanting an optimal solution to (LP), he also wants to know how sensitive the optimal solution is to changes in c_2 and c_3. He knows that the fluctuations in c_2 and c_3 are at most one unit up and down. AF is not an optimization expert, and he has considered asking an expert about assistance in setting up a stochastic programming model. Give AF a qualified advice on what to do. ....(4p)

(c) AF is also worried about b_3. He has been asked how sensitive the optimal value is to changes in b_3. Help AF to provide this information. ....(2p)

2. Consider a binary knapsack problem (KP) defined as

$$(KP) \qquad \begin{array}{ll} \text{minimize} & -\displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{subject to} & -\displaystyle\sum_{j=1}^{n} a_j x_j \geq -b, \\ & x_j \in \{0,1\}, \quad j = 1, \dots, n, \end{array}$$

where $a \geq 0$, $c \geq 0$ and $b \geq 0$.

(a) Give an explicit expression for the objective function $\varphi(\lambda)$ of the dual problem (D) arising when the constraint $-\sum_{j=1}^{n} a_j x_j \geq -b$ is relaxed by Lagrangian relaxation. ....(3p)

(b) For a given $\lambda \in I\!R$, find an explicit expression for a subgradient to φ at λ. (2p)

(c) Assume that $n = 3$, $a = (2\ 3\ 4)^T$, $b = 5$ and $c = (4\ 5\ 10)^T$. Illustrate the dual problem graphically. From the figure calculate the optimal solution and the optimal objective value of the dual problem. Solve the small (KP) by inspection, and determine the duality gap. ....(5p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 20 \end{pmatrix}, \quad c = \begin{pmatrix} 2 & 3 & 3 & 3 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior-point method. Assume further that we initially choose $x^{(0)} = (4\,3\,2\,1)^T$, $y^{(0)} = (0\,0)^T$, $s^{(0)} = (1\,2\,3\,4)^T$, and $\mu = 1$. Here, y and s denote the dual variables.

(a) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior-point method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. .... (7p)

(b) The solution to the above system of linear equations is given by

$$\Delta x \approx \begin{pmatrix} 0.4588 \\ -0.7000 \\ 0.0235 \\ 0.2176 \end{pmatrix}, \quad \Delta y \approx \begin{pmatrix} 1.5294 \\ 0.3353 \end{pmatrix}, \quad \Delta s \approx \begin{pmatrix} -0.8647 \\ -1.2000 \\ -2.5353 \\ -3.8706 \end{pmatrix}.$$

Show how these values may be used to determine $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ in a suitable way. Complete the calculations up to the point where you would need a calculator. .... (3p)

4. Consider the stochastic program (P) given by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \ldots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \ldots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded, where roll i has width w_i, $i = 1, \dots, m$. The demand for roll i is given by b_i, $i = 1, \dots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(3\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. .... (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem... (2p)

Good luck!

GAMS file for Question 1:

```
sets
i rows       / i1*i3 /
j columns    / j1*j5 /;

table A(i,j)    constraint matrix
        j1   j2   j3   j4   j5
i1       2    3    1
i2      -1    2         1
i3       2    1              1;

parameter b(i)
          / i1   4
            i2   1
            i3   5 /;

parameter c(j)
          / j1 -8
            j2  2
            j4  2
            j5 -2 /;

variables
        objval  objective function value
        x(j)    decision variables;

positive variable x;

equations
        cons(i)         constraints
        objfun          objective function;

cons(i) .. sum(j,A(i,j)*x(j)) =e= b(i);
objfun .. sum(j,c(j)*x(j)) =e= objval;

model lpex / all /;

solve lpex using lp minimizing objval;
```
