SF2812 Applied linear optimization, final exam
Monday March 13 2017 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) be defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & -1 & 0 \\ -1 & 4 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 6 & 1 & 2 & 3 & 1 \end{pmatrix}^T.$$

(a) A person named AF has used GAMS to model and solve this problem. The GAMS input file can be found at the end of the exam. Unfortunately, AF has lost the GAMS output file. He does have a partial GAMS output file, which reads:

```
---- EQU obj                 .          .          .        -1.000

  obj  objective function

---- EQU cons  constraints

      LOWER      LEVEL      UPPER     MARGINAL

i1     5.000      5.000      5.000      2.000
i2     5.000      5.000      5.000      1.000
i3     1.000      1.000      1.000     -1.000

                            LOWER      LEVEL      UPPER     MARGINAL

---- VAR objval             -INF      14.000      +INF        .

  objval  objective function value

---- VAR x  decision variables

      LOWER      LEVEL      UPPER     MARGINAL
```

```
j1      .      2.000     +INF       .
j2      .      1.000     +INF       .
j3      .        .       +INF     4.000
j4      .        .       +INF     4.000
j5      .      1.000     +INF       .
```

AF has run several versions of the problem, and he is not sure that this file is the one that corresponds to (LP). Help AF by showing that this file gives the optimal solution to (LP). .... (4p)

(b) The reason that AF has run several versions of the file is that the coefficients for x_3 and x_4 in the objective function are not very precise. Hence, in addition to wanting an optimal solution to (LP), he also wants to know how sensitive the optimal solution is to changes in c_3 and c_4. He knows that the fluctuations in c_3 and c_4 are at most two units up and down. AF is not an optimization expert, and he has considered asking an expert about assistance in setting up a stochastic programming model. Give AF a qualified advice on how to handle this particular situation, given the above obtained results. .... (4p)

(c) AF is also worried about the precise value of b_1 and wants to know how sensitive the optimal value is to changes in b_1. Help AF to provide this information. (2p)

2. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 6 & 3 & 3 & 3 \end{pmatrix}^T.$$

(a) AF has solved (PLP) by the simplex method. He has then obtained solutions $\widehat{x} = (1 \;\; 1 \;\; 0 \;\; 0)^T$, $\widehat{y} = (2 \;\; -1)^T$, and $\widehat{s} = (0 \;\; 0 \;\; 0 \;\; 3)^T$. Verify that these solutions are optimal to (PLP) and (DLP) respectively. .... (2p)

(b) AF has then implemented a primal-dual interior method in Matlab. To test his solver, he has solved the primal-dual nonlinear equations accurately for $\mu = 10^{-4}$. He has then obtained the following approximate numbers for $x(\mu)$, $y(\mu)$, and $s(\mu)$:

```
xmu' =
    0.4227    1.5773    0.5774    0.0000

ymu' =
    1.9999   -0.9999

smu' =
    0.0002    0.0001    0.0002    2.9999
```

AF has solved the equations as accurately as possible, and he is confused. The values of $y(\mu)$ and $s(\mu)$ behave as he expects, they are near $\widehat{y}$ and $\widehat{s}$ respectively, the difference being of the order of μ. However, the values of $x(\mu)$ are nowhere near $\widehat{x}$. Explain the situation to AF. Do this by using the information given in Question 2a to characterize all optimal solutions to (PLP) and show that $x(\mu)$ is in fact close to an optimal solution. ....(8p)

3. Consider a linear program in standard form

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & A_H x = b_H, & A_H \text{ is "complicating", dimension } m \times n, \\ & A_E x = b_E, & A_E \text{ is "easy"}, \\ & x \geq 0. & \end{array}$$

Assume that $\{x : A_E x = b_E, x \geq 0\}$ is bounded with extreme points v_i, $i = 1, \ldots, k$. Assume further that the problem is solved by Dantzig-Wolfe decomposition.

The master problem becomes

$$\begin{array}{ll} \text{minimize} & c^T V \alpha \\ \text{subject to} & A_H V \alpha = b_H, \\ & e^T \alpha = 1, \\ & \alpha \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{minimize} & \sum_{i=1}^k c^T v_i \alpha_i \\ \text{subject to} & \sum_{i=1}^k A_H v_i \alpha_i = b_H, \\ & \sum_{i=1}^k \alpha_i = 1, \\ & \alpha \geq 0. \end{array}$$

Here e denotes a k-dimensional vector with all components one, and $V = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix}$.

Derive the subproblem as a linear program. ....(10p)

4. Consider the integer program (IP) defined as

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -x_1 - 4x_3 - x_4 \\ \text{subject to} & -4x_1 - 7x_2 - 6x_3 - 5x_4 \geq -10, \\ & -x_1 - x_2 \geq -1, \\ & -x_3 - x_4 \geq -1, \\ & x_j \in \{0, 1\}, \quad j = 1, \ldots, 4. \end{array}$$

Assume that the constraints $-x_1 - x_2 \geq -1$ and $-x_3 - x_4 \geq -1$ are relaxed with corresponding nonnegative multipliers v_1 and v_2. Let $\varphi(v)$ denote the resulting dual objective function. Finally, let $\widehat{v} = (1\ 2)^T$.

(a) Calculate $\varphi(\widehat{v})$. The corresponding Lagrangian relaxed problem for $v = \widehat{v}$ may be solved in any way, that need not be systematic. Give all optimal solutions to the Lagrangian relaxed problem for $v = \widehat{v}$. .... (6p)

(b) Use your result of Question 4a to give two subgradients to φ at $\widehat{v}$. ....(3p)

(c) Use your result of Question 4b to show that $\widehat{v}$ is an optimal solution to the dual problem...(1p)

5. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{array}$$

Let the dimensions of the problem be such that A is an $m \times n$ matrix and let A_i denote the ith column of A.

For $i = 1, \ldots, n$, let $\mathcal{P}_i$ be a polytope given by $\mathcal{P}_i = \{v_i \in R^m : C_i^T v_i \leq d_i\}$, for given matrices C_i of dimensions $m \times n_i$ and given vectors d_i of length n_i. Each polytope $\mathcal{P}_i$ is such that $A_i \in \mathcal{P}_i$.

The reason for introducing the sets P_i is that we are interested in solving an optimization problem which is robust against uncertainties in A, given by

$$(RP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & \max_{v_i \in \mathcal{P}_i} \{v_i^T y\} \leq c_i, \quad i = 1, \ldots, n. \end{array}$$

(a) Problem (RP) looks complicated as it has a maximization function in each constraint. Use your expertise in linear programming to formulate a linear program which is equivalent to (RP). In addition, give the dual problem associated with this linear program. .... (8p)

Hint 1: For a given y, the problem

$$\begin{array}{ll} \underset{v_i \in I\!R^m}{\text{maximize}} & y^T v_i \\ \text{subject to} & C_i^T v_i \leq d_i, \end{array}$$

is a linear program.

Hint 2: Use strong duality for linear programming.

(b) If

$$C_i^T = \begin{pmatrix} I \\ -I \end{pmatrix} \quad \text{and} \quad d_i = \begin{pmatrix} A_i \\ -A_i \end{pmatrix},$$

then $\mathcal{P}_i = \{A_i\}$. Simplify the dual problem that you derived in Question 5a as much as possible for this special case. Comment on the result. .... (2p)

Good luck!

GAMS file for exercise 1:

```
sets
i rows          / i1*i3 /
j columns       / j1*j5 /;

table A(i,j)
        j1      j2      j3      j4      j5
i1       2       1      -1
i2       1       3              -1
i3      -1       4                      -1;

parameter b(i)
          / i1   5
            i2   5
            i3   1 /;

parameter c(j)
          / j1  6
            j2  1
            j3  2
            j4  3
            j5  1 /;

variables
        objval  objective function value
        x(j)    decision variables;

positive variable x;

equations
        obj             objective function
        cons(i)         constraints;

obj .. sum(j,c(j)*x(j)) =e= objval;
cons(i) .. sum(j,A(i,j)*x(j)) =e= b(i);

model lpex / all /;

solve lpex using lp minimizing objval;
```