SF2812 Applied linear optimization, final exam
Monday March 12 2018 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 37.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) and its dual (DLP) be defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ -1 & 4 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 5 \\ 6 \\ 5 \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} 3 & 0 & 3 & 3 & -1 & 0 \end{pmatrix}^T.$$

(a) A person named AF has used GAMS to model and solve this problem. AF has been told that he can solve either (LP) or (DLP) for finding the optimal solution to (LP). He has chosen to solve (DLP). The GAMS input file can be found at the end of the exam, and a partial GAMS output file reads:

```
                           LOWER     LEVEL     UPPER    MARGINAL

---- EQU obj                 .         .         .       -1.000

  obj  objective function

---- EQU cons  constraints

      LOWER     LEVEL     UPPER    MARGINAL

j1     -INF     3.000     3.000     2.000
j2     -INF       .         .       1.000
j3     -INF     2.000     3.000      .
j4     -INF     1.000     3.000      .
j5     -INF    -1.000    -1.000     1.000
j6     -INF       .         .       3.000
```

```
                              LOWER      LEVEL      UPPER     MARGINAL

---- VAR objval               -INF       5.000      +INF         .

  objval  objective function value

---- VAR y  dual variables

        LOWER      LEVEL      UPPER     MARGINAL

i1      -INF       2.000      +INF         .
i2      -INF       1.000      +INF         .
i3      -INF      -1.000      +INF         .
i4      -INF         .        +INF         .
```

The only catch is that AF does not know how to extract the optimal solution to (LP) from the GAMS output. Help AF obtain the optimal solution to (LP) from the GAMS output file. .... (4p)

(b) AF is worried about the precise value of c_1 and wants to know how sensitive the optimal value is to changes in c_1. For c_1 changed to $3+\delta$, help AF to predict k in an expression for the optimal value of the form $5+k\delta$. Do so with no calculations. .... (2p)

(c) Give bounds on δ for which the linear variation in optimal value of Question 1b is valid. The system of linear equations that arises need not be solved in a systematic way. .... (4p)

2. Consider the linear program (LP) defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & x_1 + x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 = 2, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{array}$$

(a) For a fixed positive barrier parameter μ, formulate the primal-dual system of nonlinear equations corresponding to the problem above. In addition, use the fact that the problem is small to eliminate x and s and get an equation in y only. .... (5p)

(b) The solution of the equation in y only is given by

$$y(\mu) = \frac{5-3\mu}{4} - \frac{\sqrt{1+2\mu+9\mu^2}}{4} = 1 - \mu - \mu^2 + o(\mu^2),$$

where the last equality, suitable for small positive μ, is obtained by Taylor series expansion. Make use of your results in Question 2a and the given Taylor series expansion of $y(\mu)$ to give approximate expressions for $x(\mu)$ and $s(\mu)$ that are suitable for μ small and positive. .... (3p)

(c) Calculate $\lim_{\mu\to 0} x(\mu)$ for the $x(\mu)$ you derived in Question 2b. Is this an optimal solution? Is this a basic feasible solution? Comment on the result. .... (2p)

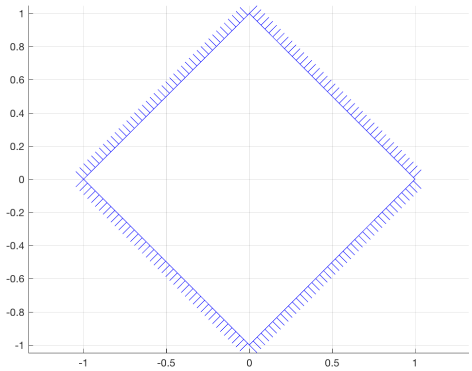
3. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{minimize} & 3x_1 + x_2 - x_3 + 3x_4 \\ \text{subject to} & 2x_1 + x_2 - x_3 - x_4 = -2, \\ & -1 \le x_1 + x_2 \le 1, \\ & -1 \le x_1 - x_2 \le 1, \\ & -1 \le x_3 + x_4 \le 1, \\ & -1 \le x_3 - x_4 \le 1. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $2x_1 + x_x - x_3 - x_4 = -2$ the complicating constraint. Use the extreme points $v_1 = (-1 \;\; 0 \;\; 1 \;\; 0)^T$ and $v_2 = (-1 \;\; 0 \;\; -1 \;\; 0)^T$ for obtaining an initial feasible solution to the master problem.

The subproblem(s) that arise may be solved in any way, that need not be systematic. .... (10p)

Hint: The following figure may be helpful:



4. Consider the integer program (IP) defined by

$$(IP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge b, \\ & Cx \ge d, \\ & x \ge 0, \quad x \text{ integer}. \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D) \quad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \ge 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \ge d, x \ge 0 \text{ integer}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integer requirement, i.e.,

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge b, \\ & Cx \ge d, \\ & x \ge 0. \end{array}$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. ....(10p)

5. Conditional value at risk (CVaR) is a risk measure used for example in optimization of radiation therapy or finance. In radiation therapy, one situation is when a sensitive organ is to be protected from too high level of radiation. The organ is divided into m volume elements, where element i has relative volume Δ_i and receives dose d_i. Relative volume means that the volume of the target is normalized so that $\sum_{i=1}^m \Delta_i = 1$. The radiation is sent through a grid of n beam elements, where the nonnegative fluence in beam element j is denoted by x_j. There is a linear relationship between x and d through a constant matrix P, so that $d = Px$.

In this setting, the conditional value at risk is defined for a fixed parameter α, with $0 < \alpha < 1$, as the optimal value of the optimization problem

$$(P_\alpha) \quad \begin{array}{ll} \underset{d\in I\!R^m, x\in I\!R^n, \lambda\in I\!R}{\text{minimize}} & \lambda + \dfrac{1}{\alpha}\displaystyle\sum_{i=1}^m \Delta_i (d_i - \lambda)_+, \\ \text{subject to} & d = Px, \quad x \geq 0. \end{array}$$

where $(d_i - \lambda)_+$ denotes $\max\{d_i - \lambda, 0\}$.

For a fixed x, d is given by $d = Px$, and we may define

$$\phi_x(\lambda) = \lambda + \frac{1}{\alpha}\sum_{i=1}^m \Delta_i (d_i - \lambda)_+.$$

(a) For simplicity of notation, assume that the volume elements are ordered so that $d_1 \geq d_2 \geq \ldots \geq d_m$.
Show that a global minimizer to $\min_\lambda\{\phi_x(\lambda)\}$ is given by $\lambda = d_{m_\alpha}$, where m_α is the smallest index j for which $\sum_{i=1}^j \Delta_i > \alpha$. .... (6p)
Hint: It may be helpful to first show that, if $\lambda \neq d_i$, $i = 1, \ldots, m$, then

$$\phi_x(\lambda) = \lambda + \frac{1}{\alpha}\sum_{i=1}^{m_\lambda} \Delta_i (d_i - \lambda) \quad \text{and} \quad \frac{d\phi_x(\lambda)}{d\lambda} = 1 - \frac{1}{\alpha}\sum_{i=1}^{m_\lambda} \Delta_i,$$

where m_λ is the largest index j for which $d_j > \lambda$.
Remark: A consequence of this result is that the optimal value of $\min_\lambda\{\phi_x(\lambda)\}$ is the average dose of the fraction α of the organ that receives the highest dose.

(b) Reformulate (P_α) as a linear program. .... (4p)
Hint: It may be helpful to introduce a new variable to be minimized, the way it would be done in GAMS.

Remark: Note that Question 5a and Question 5b can be solved independently of each other.

Good luck!

GAMS file for Question 1:

```
sets
i rows          / i1*i4 /
j columns       / j1*j6 /;

table A(i,j) values of the blocks
      j1    j2    j3   j4    j5    j6
i1     1     1     1     1
i2     2     1     1    -1
i3     1     3     1           1
i4    -1     4     1                 1 ;

parameter b(i)
  / i1   3
    i2   5
    i3   6
    i4   5 /;

parameter c(j)
  / j1  3
    j2  0
    j3  3
    j4  3
    j5 -1 /;

variables
objval objective function value
y(i)    dual variables;

equations
        obj  objective function
        cons(j)  constraints;

obj .. sum(i,b(i)*y(i)) =e= objval;
cons(j) .. sum(i,A(i,j)*y(i)) =l= c(j);

model lpex / all /;

solve lpex using lp maximizing objval;
```
