KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday December 15 2007 8.00–13.00
Brief solutions

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1. (a) We have

$$\nabla f(x) = \begin{pmatrix} 2e^{(x_1-1)} + 2x_1 - 2x_2 \\ 2x_2 - 2x_1 \\ 2x_3 \end{pmatrix} \quad \text{and in particular} \quad \nabla f(x^*) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

Since $\nabla f(x^*) \neq 0$, the point x^* does not satisfy the first-order optimality conditions for an unconstrained problem. Hence, at least one constraint must be active. The point x^* is feasible, and the only potentially active constraint is constraint 2 for $c = 2$. Since

$$\nabla g_2(x) = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^T,$$

it follows that for $c = 2$, the first-order necessary optimality conditions require a $\lambda_2 \geq 0$ such that

$$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \lambda_2,$$

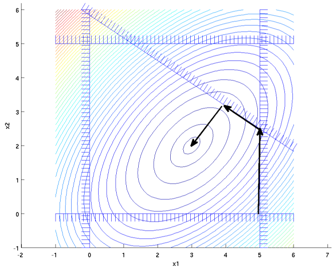
which holds for $\lambda_2 = 2$. Hence, for $c = 2$, it holds that x^* satisfies the first-order necessary optimality conditions.

(b) The objective function is convex, and the only active constraint is linear. Hence, x^* is a global minimizer to

$$(NLP') \qquad \begin{array}{ll} \text{minimize} & 2e^{(x_1-1)} + (x_2 - x_1)^2 + x_3^2 \\ \text{subject to} & x_1 + x_3 \geq 2. \end{array}$$

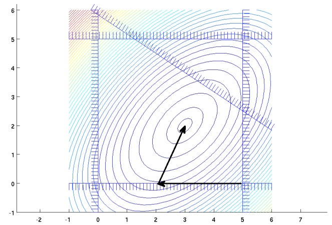
But since x^* is feasible to (NLP) as well, and the only difference between (NLP) and (NLP') is that we have omitted the constraints that are not active at x^*, it follows that x^* is globally optimal to (NLP) as well.

2. (a) The iterates are illustrated in the figure below:



At the first iteration constraint 3 is in the working set. The direction points at $(3\ 0)^T$, which is infeasible. The maximum step gives the new point $(3\ \frac{5}{2})^T$. Constraint 5 is added, which gives a vertex and hence a zero step. Constraint 3 has a negative multiplier, and it is hence deleted. The direction points at $(\frac{305}{76}\ \frac{60}{19})^T$, which is feasible. Constraint 5 has a negative multiplier, and it is hence deleted. The direction points at $(3\ 2)$ which is feasible. No constraints are active, and we have found the optimal solution.

(b) The iterates are illustrated in the figure below:



At the first iteration constraint 2 is in the working set. The direction points at $(2\ 0)^T$, which is feasible. Constraint 2 has a negative multiplier, and it is hence deleted. The direction points at $(3\ 2)$ which is feasible. No constraints are active, and we have found the optimal solution.

3. We have

$$\nabla f(x^{(0)}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g(x^{(0)}) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix},$$

$$A(x^{(0)}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \nabla^2_{xx}\mathcal{L}(x^{(0)}, \lambda^{(0)}) = \begin{pmatrix} 15 & -5 \\ -5 & 9 \end{pmatrix}.$$

(a) The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)}, \lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{då} & A(x^{(0)})\ p \geq -g(x^{(0)}). \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & \frac{15}{2}p_1^2 - 5p_1p_2 + \frac{9}{2}p_2^2 + p_1 \\ \text{då} & p_1 + p_2 \geq -2, \\ & p_2 \geq -1, \\ & p_1 \geq -4. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem.

(b) If no slack variables are added, the linear system of equations becomes

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x^{(0)}, \lambda^{(0)}) & A(x^{(0)})^T \\ \Lambda^{(0)}A(x^{(0)}) & -G(x^{(0)}) \end{pmatrix} \begin{pmatrix} \Delta x \\ -\Delta\lambda \end{pmatrix} = -\begin{pmatrix} \nabla f(x^{(0)}) - A(x^{(0)})^T\lambda^{(0)} \\ G(x^{(0)})\lambda^{(0)} - \mu e \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{pmatrix} 15 & -5 & 1 & 0 & 1 \\ -5 & 9 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ -\Delta\lambda_1 \\ -\Delta\lambda_2 \\ -\Delta\lambda_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ \mu - 2 \\ \mu - 2 \\ \mu - 12 \end{pmatrix},$$

where the value of μ has not been specified. We may for example let $\mu = 1$. We obtain $x^{(1)} = x^{(0)} + \alpha\Delta x$ and $\lambda^{(1)} = \lambda^{(0)} + \alpha\Delta\lambda$, where α is a nonnegative steplength such that $g(x^{(0)} + \alpha\Delta x) > 0$ and $\lambda^{(0)} + \alpha\Delta\lambda > 0$.

4. (See the course material.)

5. (a) The first-order optimality conditions may be written as

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ -\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \end{pmatrix}.$$

The equations may be separated as one system for x_1, x_2, λ_1, and one system for x_3, x_4. The solution is

$$x^* = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 & 1 \end{pmatrix}^T, \quad \lambda^* = \frac{3}{2}.$$

(b) To check whether x^* is a minimizer, we need to know the definiteness of $Z^T HZ$. We may compute Z as

$$Z = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{which gives} \quad Z^T HZ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

We see that $Z^T HZ$ is block diagonal with the second diagonal block indefinite. With $d = (0\ 1\ -1)$, we obtain $d^T Z^T HZd < 0$, so that H is not positive semidefinite. Hence, x^* is not a local minimizer to (EQP).

(c) Since x^* is feasible to the added constraint, it follows that a solution to the first-order optimality conditions is given by

$$x^* = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 & 1 \end{pmatrix}^T, \quad \lambda^* = \begin{pmatrix} \frac{3}{2} & 0 \end{pmatrix}^T.$$

(d) We may compute Z as

$$Z = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{which gives} \quad Z^T HZ = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}.$$

We see that $Z^T HZ$ is diagonal and positive definite. Hence, x^* is a global minimizer to (EQP). Global optimality follows since for an equality-constrained quadratic program, all local minimizers are global minimizers.