KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday May 28 2011 9.00–14.00
Brief solutions

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1. (a) Both constraints are active at x^*. The first-order necessary optimality conditions then require the existence of nonnegative λ_1^* and λ_2^* such that

$$\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \lambda_1^* + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \lambda_2^*.$$

There is a unique solution with $\lambda_1^* = 1$ and $\lambda_2^* = 2$, so that x^* satisfies the first-order necessary optimality conditions together with λ^*.

(b) Both lagrange multipliers are strictly positive, so that strict complementarity holds. A matrix $Z_+(x^*)$ whose columns form a basis for the nullspace of the matrix formed of the constraint gradients of the constraints with positive Lagrange multipliers, evaluated at x^*, is given by $Z_+(x^*) = (0\ 1\ 0)^T$. In addition to the first-order necessary optimality conditions, the second-order sufficient optimality conditions require

$$Z_+(x^*)^T \left(\nabla^2 f(x^*) - \lambda_2^* \nabla^2 g(x^*)\right) Z_+(x^*) \succ 0,$$

which gives

$$2 - 2\nabla^2 g(x^*)_{22} > 0.$$

Hence, x^* is a local minimizer if $\nabla^2 g(x^*)_{22} < 1$.

(c) Since conditions on f are only known at x^*, it is not sufficient to put any conditions on $\nabla^2 g(x)$ to ensure global minimality.

2. (See the course material.)

3. (a) In an interior method, we need to ensure that the constraint, on which the barrier transformation is applied, is satisfied with strict inequality. Hence, if the barrier is applied on $g(x) \geq 0$, we must that $g(x^{(k)}) > 0$ for all iterates k. Since $g(x^{(0)}) = -4 < 0$, some reformulation is needed.

(b) The Newton step Δx, Δs, $\Delta \lambda$ is given by

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x,\lambda) & 0 & -A(x)^T \\ A(x) & -I & 0 \\ 0 & \Lambda & S \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta \lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x) - A(x)^T\lambda \\ g(x) - s \\ S\lambda - \mu e \end{pmatrix}.$$

In our case we get

$$\begin{pmatrix} 2\lambda & 0 & 0 & 2x_1 \\ 0 & 2\lambda & 0 & 2x_2 \\ -2x_1 & -2x_2 & -1 & 0 \\ 0 & 0 & \lambda & s \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} 1 + 2x_1\lambda \\ 2x_2\lambda \\ -x_1^2 - x_2^2 + 1 - s \\ s\lambda - \mu \end{pmatrix}.$$

The initial value of s should be strictly positive. For example, let $s^{(0)} = 1/2$. Then, for the first iteration we obtain

$$\begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 4 & 0 & 4 \\ -2 & -4 & -1 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ \Delta s^{(0)} \\ \Delta\lambda^{(0)} \end{pmatrix} = - \begin{pmatrix} 5 \\ 8 \\ -\frac{9}{2} \\ 0 \end{pmatrix}.$$

(c) The next iterate is given by $x^{(1)} = x^{(0)} + \alpha^{(0)}\Delta x_1^{(0)}$, $s^{(1)} = s^{(0)} + \alpha^{(0)}\Delta s^{(0)}$, $\lambda^{(1)} = \lambda^{(0)} + \alpha^{(0)}\Delta\lambda^{(0)}$, where $\alpha^{(0)}$ is given by some approximate linesearch. The steplength $\alpha^{(0)}$ must be chosen such that $s^{(0)} + \alpha^{(0)}\Delta s^{(0)} > 0$ and $\lambda^{(0)} + \alpha^{(0)}\Delta\lambda^{(0)} > 0$.

4. The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)}, \lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g_i(x^{(0)})^T p \geq -g_i(x^{(0)}), \quad i = 1, 2, 3. \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & p_1^2 + p_2^2 \\ \text{subject to} & p_1 + p_2 \geq -2, \\ & p_1 \geq 1, \\ & p_2 \geq 1. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem.

The quadratic program is convex, and the optimal solution is given by $p^{(0)} = (1\ 1)^T$, so that $x^{(2)} = x^{(0)} + p^{(0)} = (1\ 1)^T$. The Lagrange multiplier of the quadratic program is given by$\lambda^{(1)} = (0\ 2\ 2)^T$.

5. (a) In the first iteration, constraint $x_1 + x_2 \geq -3$ is kept active. We obtain $x^{(1)} = (-1\ -2)^T$, with $\lambda_1^{(1)} = -5$. Verification gives $Hx^{(1)} + c = A_1\lambda_1^{(1)}$ and $A_1x^{(1)} = b_1$. Since $\lambda_1^{(1)} = -5 < 0$, the first constraint is deleted from the working set.

In the second iteration, no constraints are active. The search direction is given by $Hp^{(1)} = -(Hx^{(1)} + c)$, which leads towards the unconstrained minimizer. The step is limited by constraint 4, $-x_1 - x_2 \geq -3$, at the point $x^{(2)} = (1\ 2)^T$. Constraint 4 is added to the working set.

In the third iteration, we obtain a zero step to the minimizer, so that $x^{(3)} = x^{(2)} = (1\ 2)^T$. The Lagrange multiplier is given by $\lambda_4^{(3)} = 2$, so that the global minimizer has been found. Verification gives $Hx^{(3)} + c = A_4\lambda_4^{(3)}$ and $A_4x^{(3)} = b_4$.

(b) To be precise, the first statement should be that there is one iteration where the constraint $x_1 + x_2 \geq -3$ *has been* deleted and the constraint $-x_1 - x_2 \geq -3$ is added. Then, AF's claim is true. The feature which makes this happen is that we delete one constraint and add one constraint, where the gradients of these constraints are parallel (in opposite directions). In our case, $A_4 = -A_1$, constraint 1 is deleted in the first iteration and constraint 4 is added in the second iteration.

We have $Hx^{(1)} + c = A_1\lambda_1^{(1)}$, with $\lambda_1^{(1)} < 0$, and $Hp^{(1)} = -(Hx^{(1)} + c)$. Hence,

$$H(x^{(1)} + \alpha p^{(1)}) + c = Hx^{(1)} + c + \alpha Hp^{(1)}$$
$$= (1 - \alpha)(Hx^{(1)} + c) = (1 - \alpha)A_1\lambda_1^{(1)}.$$

Note that since $A_4 = -A_1$, we obtain

$$H(x^{(1)} + \alpha p^{(1)}) + c = (1 - \alpha)A_1\lambda_1^{(1)} = (1 - \alpha)A_4(-\lambda_1^{(1)}).$$

Consequently, for the value of α for which constraint $A_4x \geq b_4$ becomes active, we have a minimizer with respect to this constraint as an equality, with Lagrange multiplier $-(1-\alpha)\lambda_1^{(1)}$. As $\alpha < 1$ and $\lambda_1^{(1)} < 0$, this multiplier is positive. We are thus optimal with respect to the inequality constraint $A_4x \geq b_4$.