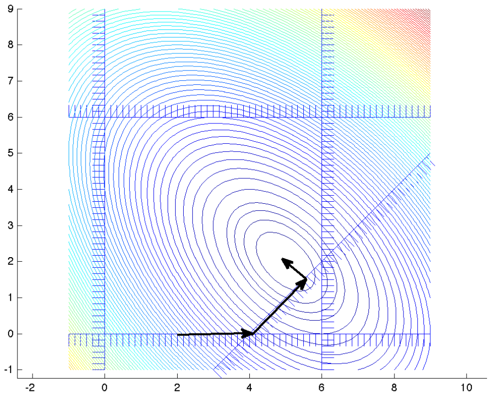
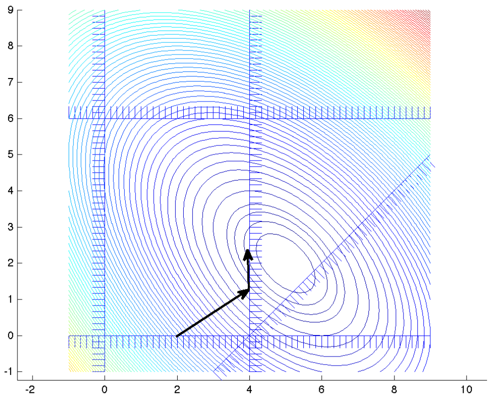
# SF2822 Applied nonlinear optimization, final exam
## Wednesday June 3 2015 14.00–19.00
## Brief solutions

**1.** (a) The iterations are illustrated in the figure below:




In the first iteration the search direction points at $(6\ 0)^T$, but the step is limited by the constraint $-x_1+x_2 \geq -4$, which is added so that the new point is $(4\ 0)^T$. A zero step is taken, and the multiplier for the constraint $x_2 = 0$ is negative, -9. This constraint is deleted. The new step points at $(11/2\ 3/2)$, which is feasible. A unit step is taken, and the multiplier for $-x_1 + x_2 = -4$ is negative, $-1/2$. This constraint is deleted. The new step points at $(5\ 2)^T$, which is feasible. No constraints are active, so this point is optimal.

(b) The iterations are illustrated in the figure below:




In the first iteration the search direction points at $(5\ 2)^T$, but the step is limited by the constraint $-x_1 \geq -4$, which is added, and the new point is $(4\ 4/3)$. The new step points at $(4\ 5/2)$, which is feasible. The multiplier of the constraint is positive, $3/2$, so that an optimal solution has been found.

**2.** We have

$$f(x) = (x_1-1)^2 + \frac{1}{2}(x_2-2)^2 \qquad g(x) = \frac{3}{2} - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \geq 0,$$

$$\nabla f(x) = \begin{pmatrix} 2(x_1-1) \\ x_2-2 \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) The point $x^{(0)}$ is not feasible.

(b) Insertion of numerical values in the expressions above gives the first QP-problem according to

$$\begin{array}{ll} \min & 2p_1^2 + \frac{3}{2}p_2^2 \\ \text{subject to} & -p_1 - 2p_2 \ge 1. \end{array}$$

This is a convex QP-problem with a globally optimal solution given by

$$\begin{aligned} 4p_1 + \lambda &= 0, \\ 3p_2 + 2\lambda &= 0, \\ -p_1 - 2p_2 &= 1. \end{aligned}$$

The solution is given by $p_1 = -3/19$, $p_2 = -8/19$ and $\lambda = 12/19$. Hence,

$$x^{(1)} = x^{(0)} + p = \begin{pmatrix} \frac{16}{19} \\ \frac{30}{19} \end{pmatrix}, \quad \lambda^{(1)} = \lambda = \frac{12}{19}.$$

**3.** Since $g(x^{(0)}) \not> 0$, we cannot use $x^{(0)}$ as an initial point without some modification. We may for example introduce a slack variable $s$ and write the primal-dual nonlinear equations as

$$\begin{aligned} \nabla f(x) - A(x)^T\lambda &= 0, \\ g(x) - s &= 0, \\ S\lambda - \mu e &= 0. \end{aligned}$$

The Newton step $\Delta x$, $\Delta s$, $\Delta\lambda$ is given by

$$\begin{pmatrix} \nabla_{xx}^2\mathcal{L}(x,\lambda) & 0 & -A(x)^T \\ A(x) & -I & 0 \\ 0 & \Lambda & S \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x) - A(x)^T\lambda \\ g(x) - s \\ S\lambda - \mu e \end{pmatrix}.$$

where $G(x) = \text{diag}(g(x))$, $\Lambda = \text{diag}(\lambda)$, $e$ is the vector of ones, $s$ is the slack variable and $S = \text{diag}(s)$. It is essential that $s^{(0)} > 0$, but we do not need to require $g(x^{(0)}) > 0$.

In our case we get

$$\begin{pmatrix} 2+\lambda & 0 & 0 & x_1 \\ 0 & 1+\lambda & 0 & x_2 \\ -x_1 & -x_2 & -1 & 0 \\ 0 & 0 & \lambda & s \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} 2(x_1-1) + x_1\lambda \\ x_2 - 2 + x_2\lambda \\ \frac{3}{2} - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - s \\ s\lambda - \mu \end{pmatrix}.$$

We may for example initialize $s$ by $s^{(0)} = 1$. Then, for the first iteration we obtain

$$\begin{pmatrix} 4 & 0 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ -1 & -2 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \\ \Delta s^{(0)} \\ \Delta\lambda^{(0)} \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 2 \\ 0 \end{pmatrix}.$$

The next iterate is given by $x^{(1)} = x^{(0)} + \alpha^{(0)} \Delta x_1^{(0)}$, $s^{(1)} = s^{(0)} + \alpha^{(0)} \Delta s^{(0)}$, $\lambda^{(1)} = \lambda^{(0)} + \alpha^{(0)} \Delta\lambda^{(0)}$, where $\alpha^{(0)}$ is given by some approximate linesearch. The steplength $\alpha^{(0)}$ must be chosen such that $s^{(0)} + \alpha^{(0)} \Delta s^{(0)} > 0$ and $\lambda^{(0)} + \alpha^{(0)} \Delta\lambda^{(0)} > 0$. As we have Newton's method, an initial estimate of steplength 1 in the approximate linesearch appears suitable.

**4.** (See the course material.)

**5.** (a) The matrix $D$ has two negative eigenvalues. Since the problem has one constraint, at most one constraint can be active at any feasble point. At least two constraints would have to be active for the reduced Hessian of the objective function with respect to the active constraints to be positive semidefinite. This is a necessary condition for a local minimizer. Hence, no local minimizer can exist.

(b) As outlined in the answer to Question 5a, since $D$ is diagonal with $D_{ii} < 0$, $i = 1, 3$, there must be at least two active constraints at any local minimizer. Hence, it must hold that $Ax^* = b$, and in addition at least one more constraint must be active, which is linearly independent from the rows of $A$. Let $x^* = (1\ 1\ 1\ 1)^T$. We have

$$\nabla f(x^*) = Dx^* + c = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Consequently, $x^*$ will satisfy the first-order necesary optimality conditions for the problem where we add $-x_2 \geq -1$, in which both constraints have Lagrange multipliers 1.

The matrix of the active constraints is then given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We now want to investigate the definiteness of $D$ on the nullspace of this matrix. However, we may then fix the second component of the vector to 0 and consider only components 1, 3 and 4. But then we have two negative eigenvalues and one constraint, so that the reduced Hessian cannot be positive semidefinite.

Consequently, it is not possible to add a bound-constraint so that $x^*$ becomes a local minimizer to the corresponding problem.