SF2822 Applied nonlinear optimization, final exam
Thursday June 1 2017 8.00–13.00
Brief solutions

1. (a) The first-order necessary optimality conditions for (QP) are given by $Hx+c=0$. As H is nonsingular, there is a unique solution given by $x^1 = (1\ 1\ 1)^T$.

The matrix H is not positive semidefinite, since the leading two-by-two principal submatrix is indefinite. With $d = (1\ \ -1\ 0)^T$, we obtain $d^T H d = -4$. Consequently, x^1 does not satisfy the second-order necessary optimality conditions to (QP).

Therefore, there is no point that satisfies the second-order necessary optimality conditions for (QP).

(b) The first-order necessary optimality conditions for (EQP) are given by

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -\lambda \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix}$$

which has unique solution $x^2 = (4\ 0\ 1)^T$, $\lambda^2 = 8$. We may for example form a matrix Z whose columns form a basis for $\text{null}(A)$ as

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

for which $Z^T H Z = I$. Hence, x^2 satisfies the second-order necessary optimality conditions fo (EQP).

(c) Since A has only one row, a local minimizer to (IQP) has to be a local minimizer to (QP) or a local minimizer to (EQP). Since x^1 does not satisfy the second-order necessary optimality conditions to (QP), it is not a local mininimizer to (QP). Hence, it is not a local minimizer to (IQP). Since x^2 satisfies the second-order sufficient optimality conditions to (EQP), it is a local minimizer to (EQP). In addition, since $\lambda^2 > 0$, it is also a local minimizer to (IQP).

(d) Let $q(x) = \frac{1}{2}x^T H x + c^T x$. With d given as in (1a), it follows that $q(x^1 + \alpha d)$ and $q(x^1 - \alpha d)$ tend to minus infinity as $\alpha \to \infty$. Since we have only one constraint, at least one of $x^1 + \alpha d$ and $x^1 - \alpha d$ must remain feasible in (IQP) as $\alpha \to \infty$. We conclude that no global minimizer can exist.

2. The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} p^T \nabla^2_{xx} \mathcal{L}(x^{(0)}, \lambda^{(0)}) p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g_i(x^{(0)})^T p \geq -g_i(x^{(0)}), \quad i = 1, 2, 3. \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 \\ \text{subject to} & p_1 \geq 1, \\ & p_2 \geq 2, \\ & p_1 + p_2 \geq 2. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem.

The quadratic program is convex, and it follows by inspection that the optimal solution is given by $p^{(0)} = (1\ \ 2)^T$. The corresponding Lagrange multipliers are given by $\lambda^{(1)} = (1\ 2\ 0)^T$. Then, $p^{(0)}$ and $\lambda^{(1)}$ satisfy the first-order necessary optimality conditions for the QP-subproblem, which by convexity gives a global minimizer. Therefore the next SQP iterate is given by $x^{(1)} = x^{(0)} + p^{(0)} = (1\ \ 2)^T$ and $\lambda^{(1)} = (1\ 2\ 0)^T$.

3. (See the course material.)

4. (a) The problem (QP) is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

$$(P_\mu) \qquad \min \ \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - \mu \ln(x_1 + x_2 - a)$$

under the implicit condition that $x_1 + x_2 - a > 0$. The first-order optimality conditions of (P_μ) gives

$$x_1(\mu) - \frac{\mu}{x_1(\mu) + x_2(\mu) - a} = 0,$$
$$x_2(\mu) - \frac{\mu}{x_1(\mu) + x_2(\mu) - a} = 0.$$

These equations are symmetric in $x_1(\mu)$ and $x_2(\mu)$. Hence, $x_1(\mu) = x_2(\mu)$. This mean that $2x_1(\mu)^2 - ax_1(\mu) - \mu = 0$, from which it follows that

$$x_1(\mu) = x_2(\mu) = \frac{a}{4} + \sqrt{\frac{a^2}{16} + \frac{\mu}{2}}.$$

The plus sign has been chosen in the square root to ensure $x_1(\mu)+x_2(\mu)-a > 0$. The dual part of the trajectory, i.e. $\lambda(\mu)$, is normally given by $\lambda_i(\mu) = \mu/g_i(x(\mu))$, $i = 1, \dots, m$. Here we only have one constraint, so

$$\lambda(\mu) = \frac{\mu}{2\left(\frac{a}{4} + \sqrt{\frac{a^2}{16} + \frac{\mu}{2}}\right) - a} = \frac{\mu}{-\frac{a}{2} + 2\sqrt{\frac{a^2}{16} + \frac{\mu}{2}}} = \frac{a}{4} + \sqrt{\frac{a^2}{16} + \frac{\mu}{2}}.$$

(b) We consider three cases: (i) $a > 0$, (ii), $a = 0$ and (iii) $a > 0$.

- $a > 0$. In this case
$$x_1(\mu) = x_2(\mu) = \lambda(\mu) = \frac{a}{4} + \sqrt{\frac{a^2}{16} + \frac{\mu}{2}} = \frac{a}{4} + \frac{a}{4}\sqrt{1 + \frac{8\mu}{a^2}}$$
As $\mu \to 0$ it follows that $x(\mu) \to (a/2\ a/2)^T$ and $\lambda(\mu) \to a/2$.
The point $x^* = (a/2\ a/2)^T$ together with $\lambda^* = a/2$ satisfies the first-order optimality conditions of (QP) and is therefore a global minimizer, since (QP) is a convex problem.
- $a = 0$. In this case
$$x_1(\mu) = x_2(\mu) = \lambda(\mu) = \sqrt{\frac{\mu}{2}}.$$
As $\mu \to 0$ it follows that $x(\mu) \to (0\ 0)^T$ and $\lambda(\mu) \to 0$.
The point $x^* = (0\ 0)^T$ together with $\lambda^* = 0$ satisfies the first-order optimality conditions of (QP) and is therefore a global minimizer, since (QP) is a convex problem.

- $a < 0$. In this case
$$x_1(\mu) = x_2(\mu) = \lambda(\mu) = \frac{a}{4} + \sqrt{\frac{a^2}{16} + \frac{\mu}{2}} = \frac{a}{4} - \frac{a}{4}\sqrt{1 + \frac{8\mu}{a^2}}$$
As $\mu \to 0$ it follows that $x(\mu) \to (0\ 0)^T$ and $\lambda(\mu) \to 0$.
The point $x^* = (0\ 0)^T$ together with $\lambda^* = 0$ satisfies the first-order optimality conditions of (QP) and is therefore a global minimizer, since (QP) is a convex problem.

5. (a) The relaxed problem is a non-convex quadratic programming problem. To obtain a lower bound of the original problem we do need to calculate a global minimizer of this non-convex relaxed problem, which in general is not computationally tractable.

(b) If we let (SDP') be the problem arising as the constraint $Y = xx^T$ is added to (SDP) we can replace Y with xx^T, which by (i) gives $\text{trace}(HY) = x^T Hx$. In addition, if $Y = xx^T$, then $y_{jj} = x_j^2$, so that the constraint $y_{jj} = x_j$ is equivalent to $x_j^2 = x_j$. Consequently, (SDP') may be written as

$$(SDP') \qquad \begin{array}{ll} \min & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & \begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & x_j^2 = x_j, \quad j = 1, \dots, n. \end{array}$$

By hint (ii) we can see that the constraint

$$\begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is always fulfilled. It follows that (SDP') may be written as

$$(SDP') \qquad \begin{array}{ll} \min & c^T x + \frac{1}{2} x^T H x \\ & x_j^2 = x_j, \quad j = 1, \dots, n. \end{array}$$

But $x_j^2 = x_j$ if and only if $x_j \in \{0, 1\}$. Hence, (SDP') and (P) are equivalent.