SF2822 Applied nonlinear optimization, final exam
Thursday August 17 2017 8.00–13.00
Brief solutions

1. As $g_3(x^*) < 0$ we must have $g_3(x) \le 0$.

Since $g_1(x^*) = 0$, $g_2(x^*) = 0$, with $\nabla g_1(x^*)$ and $\nabla g_2(x^*)$ linearly independent, it follows that x^* is a regular point. Hence, the first-order necessary optimality conditions must hold. We therefore try to find λ_1 and λ_2 such that

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \lambda_2.$$

There is a unique solution given by $\lambda_1 = 2$ and $\lambda_2 = -1$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$, we must have $g_1(x) \ge 0$ and $g_2(x) \le 0$ for the first-order necessary optimality conditions to hold.

We now investigate whether this choice gives a local minimizer. The Jacobian of the active constraints at x^* is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

As the first two columns form an invertible matrix, we may for example obtain Z from

$$Z = \begin{pmatrix} -\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} Z^T(\nabla^2 f(x^*) - \lambda_1 \nabla^2 g_1(x^*) - \lambda_2 \nabla^2 g_2(x^*))Z &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= 3, \end{aligned}$$

which is a positive definite matrix. Therefore, x^* is a regular point at which strict complementarity holds, and the second-order sufficient optimality hold. Therefore, x^* is a local minimizer.

We conclude that x^* becomes a local minimizer to (NLP) for the choice $g_1(x) \ge 0$, $g_2(x) \le 0$ and $g_3(x) \le 0$.

2. No constraints are active at the initial point. Hence, the working set is empty, i.e., $\mathcal{W} = \emptyset$. Since $H = I$ and $c = 0$, we obtain $p^{(0)} = -(Hx^{(0)} + c) = -x^{(0)}$. The maximum steplength is given by

$$\alpha_{\max} = \min_{i:a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{1}{4},$$

where the minimum is attained for $i = 1$. Consequently, $\alpha^{(0)} = 1/4$ so that

$$x^{(1)} = x^{(0)} + \alpha^{(0)} p^{(0)} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

with $\mathcal{W} = \{1\}$. The solution to the corresponding equality-constrained quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ -\lambda_1^{(2)} \end{pmatrix} = -\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

We obtain

$$p^{(1)} = \begin{pmatrix} \frac{6}{5} & -\frac{12}{5} \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{5}{6},$$

where the minimum is attained for $i = 2$. Consequently, $\alpha^{(1)} = 5/6$ so that

$$x^{(2)} = x^{(1)} + \alpha^{(1)} p^{(1)} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $\mathcal{W} = \{1, 2\}$. The solution to the corresponding equality-constrained quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ -\lambda_1^{(3)} \\ -\lambda_2^{(3)} \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(2)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T.$$

As $p^{(2)} = 0$ and $\lambda^{(3)} \geq 0$, the optimal solution has been found. Hence, $x^{(2)}$ is optimal.

3. We have

$$f(x) = 2(x_1 - 2)^2 + (x_2 - 1)^2 \qquad g(x) = 1 - x_1^2 - x_2^2 \geq 0,$$

$$\nabla f(x) = \begin{pmatrix} 4(x_1 - 2) \\ 2(x_2 - 1) \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The first QP-subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g(x^{(0)})^T p \geq -g(x^{(0)}, \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \text{minimize} & 2p_1^2 + p_2^2 \\ \text{subject to} & -4p_1 - 2p_2 \geq 2. \end{array}$$

We now utilize the fact that the subproblem is of dimension two with only one constraint. The subproblem is convex, since it is a quadratic program with positive definite Hessian. The constraint must be active, since the unconstrained minimizer $p = 0$ is infeasible. Hence, we may let $p_1 = -1/2 - p_2/2$ and minimize

$$2(\frac{1}{2} + \frac{p_2}{2})^2 + p_2^2.$$

Setting the derivative to zero gives

$$0 = 2(\frac{1}{2} + \frac{p_2}{2}) + 2p_2 = 1 + 3p_2.$$

Hence, $p_2 = -1/3$, which gives $p_1 = -1/3$. Evaluating the gradient at the optimal point of the quadratic program gives

$$\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \lambda,$$

so that $\lambda = 1/3$. Consequently, we obtain

$$x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}, \quad \lambda^{(1)} = \frac{1}{3}.$$

4. (See the course material.)

5. (a) Problem (NLP) has the form which we use in the course. The objective function is convex and the constraint functions of the inequality constraints are concave. Hence, (NLP) is a convex problem.

(b) We see by inspection that $x_1^* = 0$ and $x_2^* = \sqrt{\epsilon}$ with both constraints active. The Lagrangian function for (NLP) is given by

$$\mathcal{L}(x,\lambda) = -x_2 - (1 + \epsilon - (x_1 - 1)^2 - x_2^2)\lambda_1 - (1 + \epsilon - (x_1 + 1)^2 - x_2^2)\lambda_2,$$

so that

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{pmatrix} 2(x_1 - 1)\lambda_1 + 2(x_1 + 1)\lambda_2 \\ -1 + 2x_2\lambda_1 + 2x_2\lambda_2 \end{pmatrix}.$$

Evaluation at x^* gives

$$\nabla_x \mathcal{L}(x^*, \lambda) = \begin{pmatrix} -2\lambda_1 + 2\lambda_2 \\ -1 + 2\sqrt{\epsilon}\lambda_1 + 2\sqrt{\epsilon}\lambda_2 \end{pmatrix}.$$

The Lagrange multiplier vector λ^* is given by $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$, i.e.,

$$\begin{pmatrix} -2\lambda_1^* + 2\lambda_2^* \\ -1 + 2\sqrt{\epsilon}\lambda_1^* + 2\sqrt{\epsilon}\lambda_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\epsilon > 0$, the unique solution is given by $\lambda_1^* = \lambda_2^* = 1/(4\sqrt{\epsilon})$. Consequently,

$$x^* = \begin{pmatrix} 0 \\ \sqrt{\epsilon} \end{pmatrix}, \quad \lambda^* = \begin{pmatrix} \frac{1}{4\sqrt{\epsilon}} \\ \frac{1}{4\sqrt{\epsilon}} \end{pmatrix}.$$

(c) We see that

$$x^* \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \lambda^* \to \begin{pmatrix} \infty \\ \infty \end{pmatrix}$$

when $\epsilon \to 0$.

The Jacobian of the constraints is given by

$$A(x) = \begin{pmatrix} \nabla g_1(x)^T \\ \nabla g_2(x)^T \end{pmatrix} = \begin{pmatrix} -2(x_1 - 1) & -2x_2 \\ -2(x_1 + 1) & -2x_2 \end{pmatrix},$$

so that

$$A(x^*) = \begin{pmatrix} 2 & -2\sqrt{\epsilon} \\ -2 & -2\sqrt{\epsilon} \end{pmatrix}.$$

The rows of $A(x^*)$ are linearly independent for any $\epsilon > 0$ but they become closer and closer to linearly dependent as $\epsilon \to 0$ so that for $\epsilon = 0$ they are linearly dependent.

This is reflected in the Lagrange multipliers becoming larger and larger as $\epsilon \to 0$ so that for $\epsilon = 0$ they do not exist.