KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday December 15 2007 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and rubber; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the problem

$$(NLP)\qquad \begin{array}{ll} \text{minimize} & 2e^{(x_1-1)} + (x_2 - x_1)^2 + x_3^2 \\ \text{subject to} & x_1x_2x_3 \le 2, \\ & x_1 + x_3 \ge c, \\ & x \ge 0, \end{array}$$

where c is a constant. Let $x^* = (1\ 1\ 1)^T$.

(a) Is there any value of c such that x^* satisfies the first-order necessary optimality conditions for (NLP)? ....(6p)

(b) Is there any value of c such that x^* is a global minimizer to (NLP)? ....(4p)

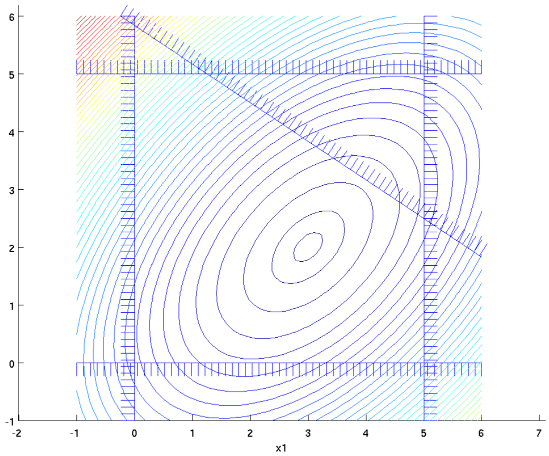
2. Consider the quadratic program (QP) defined by

$$(QP)\qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^THx + c^Tx \\ \text{subject to} & Ax \ge b, \end{array}$$

with

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -4 & -6 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -5 \\ -5 \\ -35 \end{pmatrix}.$$

The problem is illustrated geometrically in the figure below.



(a) Solve (QP) by an active-set method. Start at the point $x = (5\ \ 0)^T$ with exactly one constraint in the working set, namely $-x_1 \geq -5$. You need not compute any numerical values, but you may utilize the fact that the problem is two-dimensional and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 2a, which is appended at the end. Motivate each step carefully. .... (5p)

(b) Solve (QP) with the same method as in Exercise 2a and with the same starting point, $x = (5\ \ 0)^T$, but with $x_2 \geq 0$ as the only constraint in the working set instead. Illustrate your iterations in the figure corresponding to Exercise 2b, which is appended at the end. Motivate each step carefully. .... (5p)

3. Consider the nonlinear program

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0,\ i = 1, 2, 3, \\ & x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2, 3$, are twice-continuously differentiable. Assume specifically that we start at the point $x^{(0)} = (0\ 0)^T$ with

$$f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$g_1(x^{(0)}) = 2, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

$$g_2(x^{(0)}) = 1, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

$$g_3(x^{(0)}) = 4, \qquad \nabla g_3(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_3(x^{(0)}) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}.$$

In addition, assume that the inital estimate of the Lagrange multipliers, $\lambda^{(0)}$, are chosen as $\lambda^{(0)} = (1\ 2\ 3)^T$.

(a) Assume that we want to solve (NLP) by a sequential quadratic programming method, starting at $x^{(0)}$, $\lambda^{(0)}$. Formulate the quadratic program which is to be solved at the first iteration. Insert numerical values in your quadratic program. Show how you would use the optimal solution of the quadratic program to generate $x^{(1)}$ and $\lambda^{(1)}$. (We assume that no linesearch is needed.) .... (5p)
Note: You need *not* solve the quadratic program.

(b) Assume that we want to solve (NLP) by a primal-dual interior method, starting at $x^{(0)}$, $\lambda^{(0)}$. Formulate a linear system of equations which is to be solved at the first iteration. Insert numerical values in your linear equations. Show how you would use the solution of the linear equations to generate $x^{(1)}$ and $\lambda^{(1)}$. .... (5p)
Note: You need *not* solve the linear equations. If you choose to introduce slack variables s, assign suitable initial values to $s^{(0)}$ and show also how you would generate $s^{(1)}$.

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

4. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

5. Consider the equality-constrained quadratic program (EQP) defined by

$$(EQP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \end{array}$$

with

$$H = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ -3 \end{pmatrix}.$$

We will consider two sets of A and b.

(a) For

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \end{pmatrix},$$

compute a point that satisfies the first-order necessary optimality conditions for (EQP). .... (3p)

(b) Show that the point computed in Exercise 5a is not a local minimizer to the corresponding equality-constrained quadratic program. .... (2p)

(c) For

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

compute a point that satisfies the first-order necessary optimality conditions for (EQP). .... (2p)

(d) Show that the point computed in Exercise 5c is a global minimizer to the corresponding equality-constrained quadratic program. .... (3p)

Note: In Exercise 5, you need not solve the linear systems of equations that arise in a systematic way.

Good luck!

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Figure for Exercise 2a:

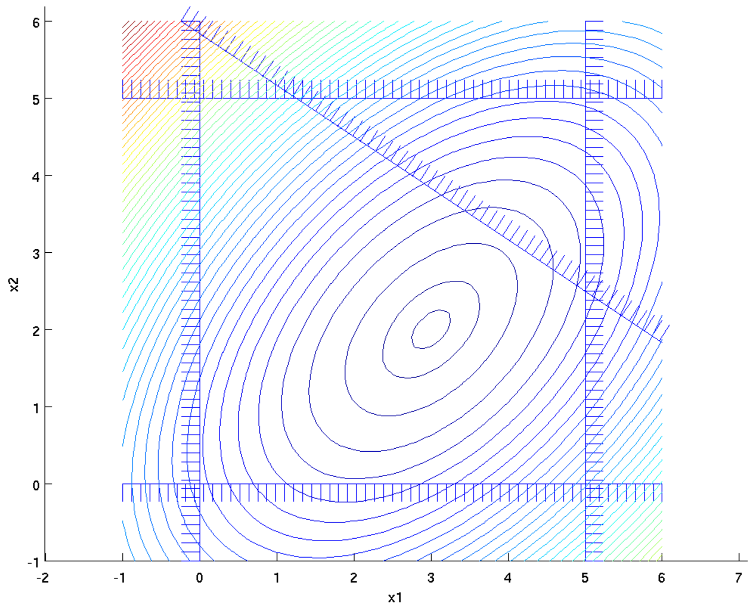

Figure for Exercise 2b: