KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday December 20 2008 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & e^{x_1} + \frac{1}{2}(x_1 + x_2 - 4)^2 + (x_1 - x_2)^2 \\ \text{subject to} & -(x_1 - 3)^2 - x_2^2 + 9 \geq 0. \end{array}$$

You have obtained a printout from a sequential quadratic programming solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Six iterations, without linesearch, have been performed. The printout, where the floating point numbers are given with four decimal places, reads:

It	x_1	x_2	λ	$\|\nabla f(x) - \nabla g(x)\lambda\|$
0	0	0	0	5.0000
1	0	1.3333	−0.7222	1.9259
2	1.0117	2.9430	−0.5596	1.1891
3	0.7084	2.1240	−0.4384	0.2028
4	0.7990	2.0422	−0.3294	0.0335
5	0.7983	2.0378	−0.3227	0.0001
6	0.7984	2.0378	−0.3227	0.0000

(a) Why cannot the above printout be correct? Give a reason that does not need any calculations in addition to the printout above. .... (2p)

(b) Formulate the first QP problem. Solve this QP problem by any method, that need not be systematic. .... (6p)

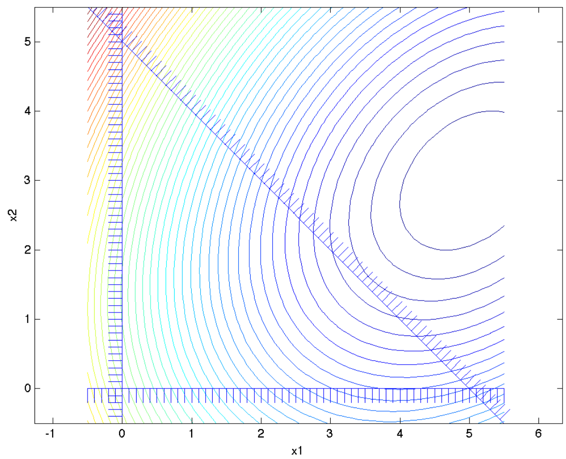
(c) The printout corresponds to a problem related to (NLP). Which one? . . (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function, where the inequality constraint is written as $g(x) \geq 0$.

2. Consider the quadratic program (QP) defined by

$$
(QP) \quad \begin{array}{ll} \text{minimize} & 3x_1^2 - 2x_1x_2 + 3x_2^2 - 24x_1 - 8x_2 \\ \text{subject to} & -x_1 - x_2 \geq -5, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}
$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (1 \ \ 0)^T$ with the constraint $x_2 \geq 0$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 2a which can be found at the last sheet of the exam. Motivate each step carefully. ....(4p)

(b) Assume that the constraint $x_2 - x_1 \geq -3$ is added to (QP), so that we obtain the problem (QP') according to

$$
(QP') \quad \begin{array}{ll} \text{minimize} & 3x_1^2 - 2x_1x_2 + 3x_2^2 - 24x_1 - 8x_2 \\ \text{subject to} & -x_1 - x_2 \geq -5, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_2 - x_1 \geq -3. \end{array}
$$

Solve (QP') by an active-set method. Start at $x = (1 \ \ 0)^T$ with the constraint $x_2 \geq 0$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Add the constraint $x_2 - x_1 \geq -3$ and illustrate your iterations in the figure corresponding to Exercise 2b which can be found at the last sheet of the exam. Motivate each step carefully. ....(6p)

3. Consider the nonlinear programming problem

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \ge 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g_i(x)\lambda_i - \mu &= 0, \quad i = 1, \ldots, m, \end{aligned}$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. .... (10p)

4. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & e^{x_1} + \frac{1}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 + x_3^2 - 2x_1 - x_2 - 3x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 + x_1 + x_2 - x_3 = 0, \\ & -x_1^2 - x_2^2 - x_3^2 \ge -2. \end{array}$$

Let $\widetilde{x} = (0 \ 0 \ 1)^T$.

(a) Show that $\widetilde{x}$ fulfils the first-order necessary optimality conditions for (NLP). .... (3p)

(b) Show that $\widetilde{x}$ fulfils the second-order sufficient optimality conditions for (NLP). .... (4p)

(c) Does it hold that $\widetilde{x}$ is a *global* minimizer to (NLP)? .... (3p)

5. For a given symmetric $n \times n$-matrix W, the so-called *two-way partitioning problem* may be posed as

$$\begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & x^T W x \\ \text{subject to} & x_{ii}^2 = 1, \quad i = 1, \ldots, n. \end{array}$$

which is a nonconvex problem in general. By Lagrangian relaxation, one may obtain the dual problem as a semidefinite program (SDP_1) on the form

$$(SDP_1) \qquad \begin{array}{ll} \underset{y \in I\!R^n}{\text{minimize}} & -e^T y \\ \text{subject to} & \text{diag}(y) \preceq W, \end{array}$$

Alternatively, one may reformulate the original problem as

$$\begin{array}{ll} \underset{X\in\mathcal{S}^n}{\text{minimize}} & \text{trace}(WX) \\ \text{subject to} & X_{ii}=1, \quad i=1,\dots,n, \\ & \text{rank}(X)=1, \quad X\succeq 0, \end{array}$$

where $\mathcal{S}^n$ denotes the set of symmetric $n\times n$ matrices. If the rank constraint is relaxed, one obtains the semidefinite program (SDP_2) given by

$$(SDP_2) \qquad \begin{array}{ll} \underset{X\in\mathcal{S}^n}{\text{minimize}} & \text{trace}(WX) \\ \text{subject to} & X_{ii}=1, \quad i=1,\dots,n, \\ & X\succeq 0. \end{array}$$

What is the relation between (SDP_1) and (SDP_2)? .... (10p)

Note: You need not consider the original problem, but you may rather consider (SDP_1) and (SDP_2) only.

Hint 1: Relate (SDP_1) and (SDP_2) via duality. Recall that a primal-dual pair of semidefinite programs are given by

$$(PSDP) \qquad \begin{array}{ll} \underset{X\in\mathcal{S}^n}{\text{minimize}} & \text{trace}(CX) \\ \text{subject to} & \text{trace}(A_iX)=b_i, \quad i=1,\dots,m, \\ & X\succeq 0, \end{array}$$

and

$$(DSDP) \qquad \begin{array}{ll} \underset{y\in\mathbb{R}^m}{\text{maximize}} & \sum_{i=1}^m b_iy_i \\ \text{subject to} & \sum_{i=1}^m A_iy_i \preceq C. \end{array}$$

Hint 2: Note that if M is a symmetric $n\times n$-matrix and u is an n-vector, then $u^TMu=\text{trace}(Muu^T)$.

Good luck!

Name: . . . . Personal number: . . . . Sheet nummer: . . . .

Figure for Exercise 2a:

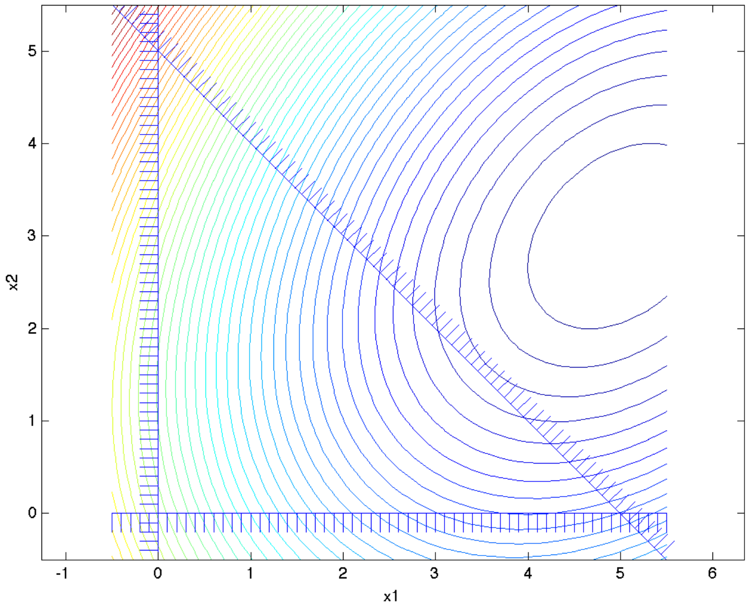


Figure for Exercise 2b: