KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Wednesday June 10 2009 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & e^{x_1} - x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 + 2x_3^2 - x_1 + x_2 - 2x_3 \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ & -x_1^2 - x_2^2 - x_3^2 \geq -2, \\ & x_2 \geq 0. \end{array}$$

Let $\widetilde{x} = (0\ 0\ 1)^T$ and let $\widehat{x} = (1\ 0\ 0)^T$.

Use first- and second-order optimality conditions for (NLP) to determine if $\widetilde{x}$ and/or $\widehat{x}$ are local minimizers of (NLP). . . . . (10p)

2. Consider the nonlinear programming problem (NLP) defined by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1 + x_2)^2 + \frac{5}{2}x_1 - \frac{1}{2}x_2 \\ \text{subject to} & x_1 \cdot x_2 - 1 \geq 0. \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)} = (2\ \ \frac{1}{2})^T$, $\lambda^{(0)} = (1\ 0\ 0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. . . . . (10p)

Note: According to the convention of the textbook we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ is the objective function and $g(x)$ is the constraint function, with the inequality constraints written as $g(x) \geq 0$.

3. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

4. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \end{array}$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \qquad c = \begin{pmatrix} -3 \\ -7 \\ -10 \\ -10 \\ -3 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 3 \\ 4 \\ 1 \end{pmatrix}$$

The optimal solution to (QP) is given by $x^* = (5 \;\; 4 \;\; 3 \;\; 2 \;\; 1)^T$.

(a) Determine a matrix Z whose columns form a basis for the nullspace of A. (2p)

(b) It turns out that c_1 was not correctly given in the original problem. It should have been $c_1 = -30$. Call this new problem (QP_2). Solve (QP_2) making use of x^* and Z. .... (4p)

(c) It turns out that in addition to $c_1 = -30$, there should have been constraints $x \geq 0$. Call this new problem (QP_3). Solve (QP_3) making use of your calculations in Exercise 4b. .... (4p)

5. Consider the optimization problem

$$(NLP) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & \frac{1}{2}\sum_{i \in \mathcal{U}} (p_i^T x - u_i)_+^2 + \frac{1}{2}\sum_{i \in \mathcal{L}} (l_i - p_i^T x)_+^2, \\ \text{subject to} & x \geq 0, \end{array}$$

where $\mathcal{L}$ and $\mathcal{U}$ are nonintersecting index sets such that $\mathcal{L} \bigcup \mathcal{U} = \{1, \ldots, m\}$, and the subscript $''+''$ denotes the positive part, i.e., $x_+ = \max(x, 0)$. The constants u_i, $i \in \mathcal{U}$, and l_i, $i \in \mathcal{L}$, are known as well as the constant vectors p_i, $i = 1, \ldots, m$. This means that we pay a quadratic penalty cost for violating lower bounds l_i, $i \in \mathcal{L}$, and upper bounds u_i, $i \in \mathcal{U}$, respectively.

The formulation (NLP) is straightforward, but a drawback is that the objective function is not twice-continuously differentiable. Your task is to show that we may obtain a smooth problem by introducing additional variables and constraints.

(a) Show that the objective function of (NLP) has continuous gradient but discontinuities in the Hessian. .... (2p)

(b) Show that (NLP) is equivalent to the quadratic programming problem

$$(QP) \qquad \begin{array}{ll} \underset{x\in I\!R^n, y\in I\!R^m}{\text{minimize}} & \frac{1}{2}\sum_{i\in\mathcal{U}} y_i^2 + \frac{1}{2}\sum_{i\in\mathcal{L}} y_i^2, \\ \text{subject to} & y_i \ge p_i^T x - u_i, \ i \in \mathcal{U}, \\ & y_i \ge l_i - p_i^T x, \ i \in \mathcal{L}, \\ & x \ge 0. \end{array}$$

Do so by showing minimization over y in (QP) for a given x gives $y_i = (p_i^T x - u_i)_+$, $i \in \mathcal{U}$, and $y_i = (l_i - p_i^T x)_+$, $i \in \mathcal{L}$. .... (4p)

(c) For a given positive barrier parameter μ, formulate the perturbed first-order optimality conditions that are to be solved approximately if a primal-dual interior method is applied to (QP). .... (4p)

Note: The motivation for considering this reformulation is that we obtain a smooth problem. The increased dimensionality introduced by the y variables can be eliminated in the linear equations that are solved in a primal-dual interior method.

Good luck!