KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Thursday December 17 2009 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the quadratic program (QP) defined by

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \\ \text{subject to} & 2x_1 + x_2 + x_3 \geq 2, \\ & x_1 + 2x_2 + x_3 \geq 3, \\ & x_1 + x_2 + 2x_3 \geq 4. \end{array}$$

Solve (QP) with an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (0\ 1\ 2)^T$. The linear equations that arise may be solved in any way, that need not be systematic. .... (10p)

2. Consider the same quadratic program (QP) as in Exercise 1, i.e.,

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \\ \text{subject to} & 2x_1 + x_2 + x_3 \geq 2, \\ & x_1 + 2x_2 + x_3 \geq 3, \\ & x_1 + x_2 + 2x_3 \geq 4. \end{array}$$

Assume that we want to solve (QP) with a primal-dual interior point method. Also assume that we initially choose $x^{(0)} = (0\ 1\ 2)^T$, $\lambda^{(0)} = (1\ 2\ 3)^T$, and $\mu = 1$.

(a) When the constraints are in the form $Ax \geq b$, one may introduce slack variables s and rewrite the constraints as $Ax - s = b$, $s \geq 0$, when applying the interior method. Explain why this is not necessary for the given initial value $x^{(0)}$. (2p)

(b) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior point method for the given initial values. Formulate the general form and then introduce explicit numerical values into the system of equations. .... (5p)

(c) If the linear system of equations of Exercise 2b are solved, and the steps in the x-direction and the λ-direction are denoted by Δx and $\Delta\lambda$ respectively, one obtains

$$\Delta x \approx \begin{pmatrix} 0.7244 \\ 0.0157 \\ -0.3386 \end{pmatrix}, \quad \Delta\lambda \approx \begin{pmatrix} -1.1260 \\ -1.8346 \\ -2.1890 \end{pmatrix}, \quad A\Delta x \approx \begin{pmatrix} 1.1260 \\ 0.4173 \\ 0.0630 \end{pmatrix}.$$

Explain why it is not suitable to use the unit step, i.e, why it is not suitable to let $x^{(1)} = x^{(0)} + \Delta x$ and $\lambda^{(1)} = \lambda^{(0)} + \Delta\lambda$. Also explain how you would choose $x^{(1)}$ and $\lambda^{(1)}$. You need not give precise numerical values of $x^{(1)}$ and $\lambda^{(1)}$, but you should explain the principle. .... (3p)

3. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

4. Consider the nonlinear program (NLP) given by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) \geq 0, \\ & x \geq 0, \\ & x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $h : I\!R^2 \to I\!R$ are twice-continuously differentiable functions, with f and $-h$ convex on $I\!R^2$.

Assume that $\widetilde{x} = (5\ 4)^T$ is a local minimizer to (NLP) with corresponding Lagrange multiplier vector $\tilde{\lambda} = (2\ 0\ 0)^T$. The notation of the coursebook is used, so that with $g_1(x) = h(x)$, $g_2(x) = x_1$ and $g_3(x) = x_2$, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

It is known that

$$f(\widetilde{x}) = 5, \qquad \nabla f(\widetilde{x}) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \qquad \nabla^2 f(\widetilde{x}) = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix},$$

$$h(\widetilde{x}) = 0, \qquad \nabla h(\widetilde{x}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \nabla^2 h(\widetilde{x}) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

It turns out that the problem was not correctly posed, but that the correct problem is (NLP') given by

$$(NLP') \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) - \frac{1}{2} \geq 0, \\ & x \geq 0, \\ & x \in I\!R^2, \end{array}$$

i.e., the first constraint has been changed from $h(x) \geq 0$ to $h(x) - \frac{1}{2} \geq 0$.

(a) Give an estimate of the optimal value of (NLP') based on the knowledge of the solution of (NLP) and corresponding Lagrange multiplier vector. .... (2p)

(b) Assume that sequential quadratic programming is used for solving (NLP'). It is then natural to let $x^{(0)} = \widetilde{x}$ and let $\lambda^{(0)} = \tilde{\lambda}$. Perform one iteration, i.e. calculate $x^{(1)}$ and $\lambda^{(1)}$ given these values of $x^{(0)}$ and $\lambda^{(0)}$. (We assume that no line search is needed). The quadratic programming problem that arises may be solved in any way, that need not be systematic. .... (8p)

5. Consider the nonlinear optimization problems (NLP_1) and (NLP_2) defined as

$$(NLP_1) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \\ & x \in I\!R^n, \end{array}$$

and

$$(NLP_2) \qquad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z - f(x) \geq 0, \\ & g(x) \geq 0, \\ & x \in I\!R^n, \ z \in I\!R, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are twice-continuously differentiable.

Assume that $x^* \in I\!R^n$ together with $\lambda^* \in I\!R^m$ satisfy the second-order sufficient optimality conditions for (NLP_1).

Determine suitable values of z and Lagrange multipliers associated with the constraints $z - f(x) \geq 0$ and $g(x) \geq 0$ so that $x = x^*$ together with these determined values of z and the Lagrange multipliers of $z - f(x) \geq 0$ and $g(x) \geq 0$ satisfy the second-order sufficient optimality conditions for (NLP_2). Verify that these optimality conditions hold. .... (10p)

Good luck!