KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Wednesday June 9 2010 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the inequality-constrained quadratic program (IQP) defined by

$$(IQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^THx + c^Tx \\ \text{subject to} & Ax \ge b, \end{array}$$

with

$$H = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ -3 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \end{pmatrix}.$$

In this exercise, you may base your arguments on the fact that the problem has only one constraint. The linear systems of equations that may arise need not be solved in a systematic way.

(a) Consider the unconstrained quadratic program

$$(QP) \qquad \text{minimize} \quad \frac{1}{2}x^THx + c^Tx.$$

Is there a point that satisfies the second-order necessary optimality conditions for (QP)? .... (3p)

(b) Consider the equality-constrained quadratic program

$$(EQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^THx + c^Tx \\ \text{subject to} & Ax = b. \end{array}$$

Is there a point that satisfies the second-order necessary optimality conditions for (EQP)? .... (3p)

(c) Does (IQP) have a local minimizer? .... (2p)

(d) Does (IQP) have a global minimizer? .... (2p)

2. Consider the quadratic program (QP) defined by

$$(QP)\quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & 2x_1 + x_2 \geq 3, \\ & x_1 + 2x_2 \geq 3. \end{array}$$

Solve (QP) with an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (5\ 0)^T$. The equality-constrained quadratic programs that arise need not be solved in a systematic way. They may for example be solved graphically. However, the values of the generated iterates x_k and corresponding Lagrange multipliers λ_k should be calculated. .... (10p)

3. Consider the nonlinear programming problem (NLP) defined by

$$(NLP)\quad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1 + x_2)^2 + \frac{5}{2}x_1 - \frac{1}{2}x_2 \\ \text{subject to} & x_1 \cdot x_2 - 1 \geq 0. \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{array}$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)} = (\frac{1}{2}\ \ 2)^T$, $\lambda^{(0)} = (1\ 0\ 0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. .... (10p)

Note: According to the convention of the textbook we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ is the objective function and $g(x)$ is the constraint function, with the inequality constraints written as $g(x) \geq 0$.

4. Consider the semidefinite programming problem (P) defined as

$$(P)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & G(x) \succeq 0, \end{array}$$

where $G(x) = \sum_{j=1}^{n} A_j x_j - B$ for B and A_j, $j = 1, \ldots, n$, are symmetric $m \times m$-matrices. The corresponding dual problem is given by

$$(D)\quad \begin{array}{ll} \text{maximize} & \operatorname{trace}(BY) \\ \text{subject to} & \operatorname{trace}(A_j Y) = c_j, \quad j = 1, \ldots, n, \\ & Y = Y^T \succeq 0. \end{array}$$

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad c^T x - \mu \ln(\det(G(x))).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} c_j - \operatorname{trace}(A_j Y) &= 0, \quad j = 1, \dots, n, \\ G(x)Y - \mu I &= 0, \end{aligned}$$

assuming that $G(x) \succ 0$ and $Y \succ 0$ are kept implicitly. .... (5p)

(b) Show that a solution $x(\mu)$ and $Y(\mu)$ to the system of nonlinear equations, such that $G(x(\mu)) \succ 0$ and $Y(\mu) \succ 0$, is feasible to (P) and (D) respectively with duality gap $m\mu$. .... (3p)

(c) In linear programming, when $G(x)$ and Y are diagonal, it is not an issue how the equation $G(x)Y - \mu I = 0$ is written. The linearizations of $G(x)Y - \mu I = 0$ and $YG(x) - \mu I = 0$ are then identical. Explain why this is in general not the case for semidefinite programming and how it can be handled. .... (2p)

Remark: For a symmetric matrix M we above use $M \succ 0$ and $M \succeq 0$ to denote that M is positive definite and positive semidefinite respectively. You may use the relations

$$\frac{\partial \ln(\det(G(x)))}{\partial x_j} = \operatorname{trace}(A_j G(x)^{-1}) \quad \text{for} \quad j = 1, \dots, n,$$

without proof.

5. Consider the optimization problem

$$(P) \qquad \underset{x \in I\!R^n}{\text{minimize}} \; \{ \max_{i=1,\dots,n} f_i(x) \},$$

where the functions f_i, $i = 1, \dots, n$, are twice continuously differentiable and convex on $I\!R^n$.

(a) For a given positive barrier parameter μ, show that we may associate a logarithmic barrier transformation with (P) that gives a problem on the form

$$(NLP_\mu) \qquad \begin{array}{ll} \text{minimize} & z - \mu \sum_{i=1}^n \ln(z - f_i(x)) \\ \text{subject to} & x \in I\!R^n, z \in I\!R, \end{array}$$

with the additional implicit constraints $z - f_i(x) > 0$, $i = 1, \dots, n$. ...... (5p)
Hint: First rewrite (P) as an equivalent nonlinear program.

(b) Derive the primal-dual nonlinear equations that correspond to a global minimizer of (NLP_μ). Motivate global minimality. .... (5p)

Good luck!