KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Saturday May 28 2011 9.00–14.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear program

$$
(NLP) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & a^T x - b \geq 0, \\ & g(x) \geq 0, \\ & x \in I\!R^3, \end{array}
$$

where $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R$, are twice-continuously differentiable. For $x^* = (1\ 1\ 1)^T$, it is known that

$$
f(x^*) = 0, \quad \nabla f(x^*) = \begin{pmatrix} -1 & 0 & 2 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

$$
b = 1, \qquad a = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T,
$$

$$
g(x^*) = 0, \quad \nabla g(x^*) = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T.
$$

(a) Does x^* satisfy the first-order necessary optimality conditions for (NLP)? (3p)

(b) Are there conditions on $\nabla^2 g(x^*)$, which guarantee that x^* is a local minimizer to (NLP)? If so, which conditions? . . . . (4p)

(c) Are there conditions on $\nabla^2 g(x)$, $x \in I\!R^3$, which guarantee that x^* is a global minimizer to (NLP)? If so, which conditions? These conditions should not include any properties related to f, a or b. . . . . (3p)

2. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

3. Consider the nonlinear program (NLP) given by

$$
(NLP) \quad \begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & -x_1^2 - x_2^2 + 1 \geq 0, \\ & x \in I\!R^2. \end{array}
$$

Assume that we want to solve (NLP) using a primal-dual interior method. Also assume that we initially choose $x^{(0)} = (1\ 2)^T$, $\lambda^{(0)} = 2$, and $\mu = 1$.

(a) When the constraints are in the form $g(x) \geq 0$, one may introduce slack variables s and rewrite the constraints as $g(x) - s = 0$, $s \geq 0$, when applying the interior method. Explain why the constraint $-x_1^2 - x_2^2 + 1 \geq 0$ needs to be rewritten in (NLP) for the given initial value $x^{(0)}$. .... (2p)

(b) Formulate the system of linear equations to be solved in the first iteration of the primal-dual interior point method where the constraint $g(x) \geq 0$ is rewritten as $g(x) - s = 0$, $s \geq 0$. Use the given $x^{(0)}$ and μ. Select an appropriate value of $s^{(0)}$. Formulate the general form and then introduce explicit numerical values into the system of linear equations. .... (6p)

(c) Assume that you have solved the system of linear equations that was formulated in Exercise 3b, so that you have obtained numerical values of its solution $\Delta x^{(0)}$, $\Delta s^{(0)}$, $\Delta\lambda^{(0)}$. Explain how you would choose $x^{(1)}$, $s^{(1)}$ and $\lambda^{(1)}$. .... (2p)

4. Consider the nonlinear program

$$
(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \ i = 1, 2, 3, \\ & x \in I\!R^2, \end{array}
$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2, 3$, are twice-continuously differentiable. Assume specifically that $x^{(0)} = (0\ 0)^T$, at which it holds that

$$
f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
g_1(x^{(0)}) = 2, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},
$$

$$
g_2(x^{(0)}) = -1, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},
$$

$$
g_3(x^{(0)}) = -1, \qquad \nabla g_3(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_3(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.
$$

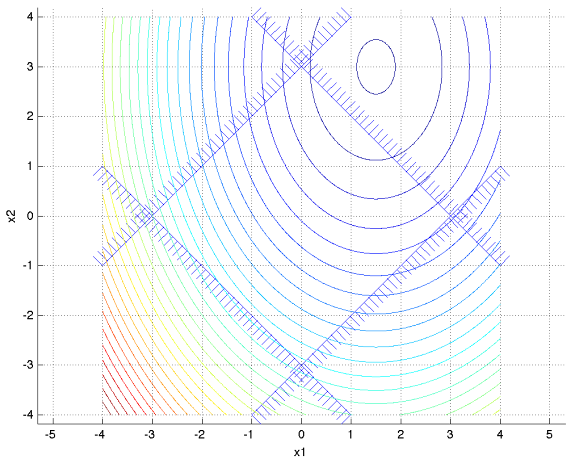
We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)}$ given above, let $\lambda^{(0)} = (1\ 0\ 0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. . . . . (10p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

5. Consider the quadratic program (QP) defined by

$$
(QP) \quad \begin{array}{ll} \text{minimize} & x_1^2 + \frac{1}{2}x_2^2 - 3x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \geq -3, \\ & x_1 - x_2 \geq -3, \\ & -x_1 + x_2 \geq -3, \\ & -x_1 - x_2 \geq -3. \end{array}
$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (-2\ \ -1)^T$ with the constraint $x_1 + x_2 \geq -3$ active. You may utilize the fact that the problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 5a which can be found at the last sheet of the exam. Verify the correctness of the iterates that you find algebraically. Motivate each step carefully. . . . . (5p)

(b) Your friend AF claims that in the active-set method of Exercise 5a, there is one iteration where the constraint $x_1 + x_2 \geq -3$ is deleted and the constraint $-x_1 - x_2 \geq -3$ is added. Then, the minimizer with respect to the inequality constraint $-x_1 - x_2 \geq -3$ has been found without taking any additional step. Is AF right? If so, which property of (QP) makes this happen? Explain algebraically. (5p)

Good luck!

Name: .... Personal number: .... Sheet nummer: ....

Figure for Exercise 5a: