KTH Mathematics

# SF2822 Applied nonlinear optimization, final exam
## Tuesday August 23 2011 14.00–19.00

*Examiner:* Anders Forsgren, tel. 790 71 27.

*Allowed tools:* Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

*Solution methods:* Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

*Note!* Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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**1.** Consider the quadratic program $(QP)$ given by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}.$$

(a) Solve $(QP)$ by an active-set method. Start at $x = (2 \;\; 5)^T$ with the constraint $-x_2 \geq -5$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 1a which can be found at the last sheet of the exam. Motivate each step carefully. ........................................................................(5p)

(b) Solve $(QP)$ again by an active-set method. This time, start at $x = (3 \;\; 4)^T$ with the constraint $-x_1 \geq -3$ active. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 1b which can be found at the last sheet of the exam. Motivate each step carefully. ........................................................(5p)

**2.** Consider the nonlinear program $(P)$ given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1-3)^2 + \frac{1}{2}(x_2-2)^2 \\ \text{subject to} & 2 - x_1^2 - \frac{1}{2}x_2^2 \ge 0. \end{array}$$

Let $x^{(0)} = (1\ 0)^T$ and let $\lambda^{(0)} = 2$. (In accordance to the notation of the textbook, we define the Lagrangian as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$.)

Assume that one wants to solve $(P)$ by a primal-dual interior method. Let $\mu = 2$ and set up the linear system of equations to be solved at the first iteration for the given $x^{(0)}$, $\lambda^{(0)}$ and $\mu$. First give the linear equations on general form and then insert numerical values. You need not solve the linear equations, but state how you would use the solution to generate then next iterate $x^{(1)}$, $\lambda^{(1)}$. . . . . . . . . . . . . . . . . . . . . . . (10p)

**3.** Consider the convex quadratic program $(QP)$ defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \end{array}$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 \\ -2 \\ -3 \\ -2 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

A feasible solution to $(QP)$ is given by $\bar{x} = (1\ \ 0\ \ 0\ \ 1)^T$.

(a) Determine a matrix $Z$ whose columns form a basis for the nullspace of $A$. (2p)

(b) Solve $(QP)$ making use of $\bar{x}$ and $Z$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (5p)

(c) Is it possible to remove one of the constraints of $(QP)$ such that the optimal solution remains unchanged? If so, which constraint? Motivate the answer carefully. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (3p)

**4.** Consider the problem $(P)$ defined as

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) = 0, \\ & x \in I\!R^n, \end{array}$$

where $x \in I\!R^n$ and $g(x) \in I\!R^m$. The first-order necessary optimality conditions of $(P)$ at a regular point take the form

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g(x) &= 0. \end{aligned}$$

Assume that Newton's method is used to solve this system of nonlinear equations in $x \in I\!R^n$ and $\lambda \in I\!R^m$. Derive the linear system of equations that needs to be solved for a given iterate $(x_k, \lambda_k)$, at a Newton iteration. Show that this system of linear equations under suitable assumptions is equivalent to a certain QP problem. Give the assumptions and state the QP problem. ................................(10p)

**5.** Consider the indefinite quadratic program $(QP)$ defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T D x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ -2 \\ 4 \\ -4 \\ 7 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

You may throughout use the fact that $D$ is diagonal.

(a) Determine how many negative eigenvalues $D$ has. Also determine the maximum number of active constraints $(QP)$ may have at any feasible point. Use these two facts to motivate why $(QP)$ cannot have any local minimizer. .......(3p)

(b) Is it possible to add a bound-constraint to $(QP)$ such that $(1\ 1\ 1\ 1\ 1)^T$ is a local minimizer to the resulting problem? If so, determine such a constraint. ......................................................................(7p)
*Note:* A bound-constraint is a constraint on the form $x_i \geq l_i$ or $-x_i \geq -u_i$, for some $i$, $i = 1, \ldots, 5$, where $l_i$ or $u_i$ is a given numeric value.

*Good luck!*

Name: . . . . . . . . . . . . . . . . Personal number: . . . . . . . . . . . . . . . . Sheet nummer: . . . . . . . . . . . . . . . .

Figure for Exercise 1a:

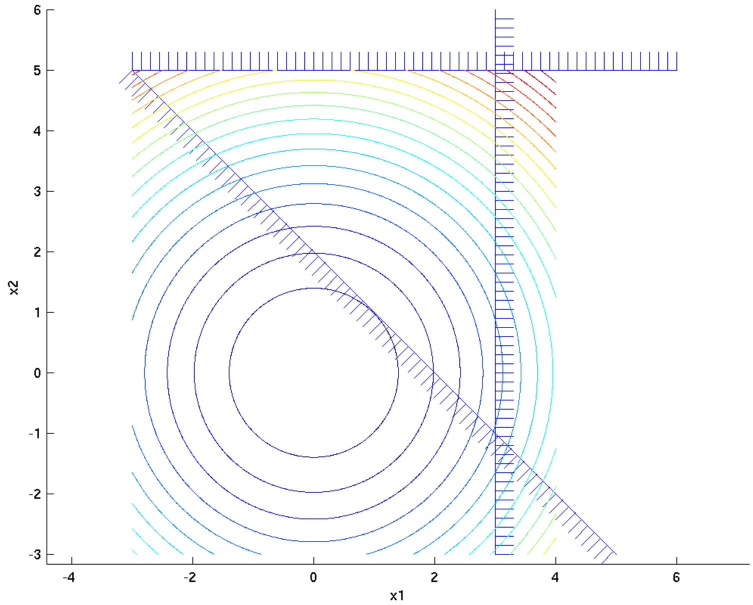


Figure for Exercise 1b: