KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Friday August 17 2012 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear programing problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \ge 0, \\ & x \in I\!R^3, \end{array}$$

where $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R$ are twice continuously differentiable.

Assume that we have a point x^* such that

$$f(x^*) = 5, \quad \nabla f(x^*) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g(x^*) = 0, \quad \nabla g(x^*) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \quad \nabla^2 g(x^*) = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Show that x^* is not a local minimizer to (NLP). .... (2p)

(b) Is it possible to add a bound-constraint to (NLP) such that x^* is a local minimizer to the resulting problem? If so, determine such a constraint.... (8p)

Note: A bound-constraint is a constraint on the form $x_i \ge l_i$ or $-x_i \ge -u_i$, for some i, $i = 1, \ldots, 3$, where l_i or u_i is a given numeric value.

2. Consider the quadratic program (QP) given by

$$(QP) \qquad \begin{array}{ll} \min & \frac{1}{2}x^THx \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \; b = \begin{pmatrix} -4 \\ -4 \\ 2 \end{pmatrix}.$$

(a) Solve (QP) using an active-set method. Start at the point $x = (2 \;\; 4)^T$ and let the constraint $-x_2 \geq -4$ be active in the first iteration. You may use the fact that the problem is two-dimensional and for example illustrate the iterations in a figure. Motivate each step carefully. .... (5p)

(b) Solve (QP) using an active-set method. Start at the point $x = (2 \;\; 4)^T$ and let no constraints be active in the first iteration. You may use the fact that the problem is two-dimensional and for example illustrate the iterations in a figure. Motivate each step carefully. .... (5p)

3. Consider the nonlinear optimization problem (NLP) given by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & 4(x_1-2)^2 + (x_2-1)^2 \\ \text{subject to} & 1 - x_1^2 - x_2^2 \geq 0. \end{array}$$

We want to solve (NLP) by a primal-dual interior method.

(a) Let $x^{(0)} = (0 \; 0)^T$ and let $\lambda^{(0)} = 1$. Let the barrier parameter $\mu = 1$. Formulate a suitable linear system of equations to be solved in the first iteration of the primal-dual interior point method for the given initial values. Formulate the general form and then introduce explicit numerical values into the linear system of equations. .... (8p)

(b) In your primal-dual method, how would you ensure that $x^{(1)}$ and $\lambda^{(1)}$ remain interior? .... (2p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

4. Consider the semidefinite programming problem (P) defined as

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^Tx \\ \text{subject to} & G(x) \succeq 0, \end{array}$$

where $G(x) = \sum_{j=1}^{n} A_j x_j - B$ for B and A_j, $j = 1, \ldots, n$, are symmetric $m \times m$-matrices. The corresponding dual problem is given by

$$(D) \quad \begin{array}{ll} \text{maximize} & \operatorname{trace}(BY) \\ \text{subject to} & \operatorname{trace}(A_j Y) = c_j, \quad j = 1, \ldots, n, \\ & Y = Y^T \succeq 0. \end{array}$$

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad c^T x - \mu \ln(\det(G(x))).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} c_j - \operatorname{trace}(A_j Y) = 0, &\quad j = 1, \ldots, n, \\ G(x)Y - \mu I = 0, & \end{aligned}$$

assuming that $G(x) \succ 0$ and $Y \succ 0$ are kept implicitly. .... (5p)

(b) Show that a solution $x(\mu)$ and $Y(\mu)$ to the system of nonlinear equations, such that $G(x(\mu)) \succ 0$ and $Y(\mu) \succ 0$, is feasible to (P) and (D) respectively with duality gap $m\mu$. .... (3p)

(c) In linear programming, when $G(x)$ and Y are diagonal, it is not an issue how the equation $G(x)Y - \mu I = 0$ is written. The linearizations of $G(x)Y - \mu I = 0$ and $YG(x) - \mu I = 0$ are then identical. Explain why this is in general not the case for semidefinite programming and how it can be handled. .... (2p)

Remark: For a symmetric matrix M we above use $M \succ 0$ and $M \succeq 0$ to denote that M is positive definite and positive semidefinite respectively. You may use the relations

$$\frac{\partial \ln(\det(G(x)))}{\partial x_j} = \operatorname{trace}(A_j G(x)^{-1}) \quad \text{for} \quad j = 1, \ldots, n,$$

without proof.

5. Consider the nonlinear programming problem (NLP) defined as

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, \ldots, m, \\ & x \in I\!R^n, \end{array}$$

where f and g are twice-continuously differentiable.

(a) Assume that (NLP) has at least one feasible point. Further, assume that we want to solve (NLP) by sequential quadratic programming. Show that if g_i, $i = 1, \ldots, m$, are concave functions on $I\!R^n$, then the quadratic programming subproblem is always feasible. .... (6p)

(b) Show that if $n = 1$, $m = 3$, $g_1(x) = x^2 - 1$, $g_2(x) = x$, $g_3(x) = -x + 2$, then (NLP) is feasible, but the first quadratic programming subproblem is infeasible if $x^{(0)} = 0$. . . . .(4p)

Good luck!