KTH Mathematics

SF2822 Applied nonlinear optimization, final exam
Monday May 20 2013 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} -5 \\ -5 \\ -5 \\ 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

A feasible solution to (QP) is given by $\bar{x} = (-1 \;\; 0 \;\; 0 \;\; 0)^T$.

(a) Show that H is a positive definite matrix. ....(1p)
Hint: It holds that $H = I + ee^T$, where I is the identity matrix and e is the vector of ones.

(b) Determine a matrix Z whose columns form a basis for the nullspace of A. (2p)

(c) Solve (QP) making use of $\bar{x}$ and Z. ....(5p)

(d) Is it possible to remove one or several of the constraints of (QP) such that the optimal solution remains unchanged? If so, which constraint or constraints? Motivate the answer carefully. .... (2p)

2. Consider the quadratic program (QP) defined by

$$(QP)\quad \begin{array}{lll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 & \\ \text{subject to} & x_1 + x_2 & \geq 6, \\ & x_1 & \geq 2, \\ & x_2 & \geq 0. \end{array}$$

Solve (QP) by an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (8\ 0)^T$ and the constraint $x_2 \geq 0$ in the working set. The equality-constrained quadratic programs that arise need not be solved in a systematic way. They may for example be solved graphically. However, the values of the generated iterates $x^{(k)}$ and corresponding Lagrange multipliers $\lambda^{(k)}$ should be calculated. .... (10p)

3. Consider the quadratic program (QP) given by

$$(QP)\quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 - 1 \geq 0. \end{array}$$

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytic expression for this small problem. ....(5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in (3a) converge to the optimal solution x^* and Lagrange multiplier λ^* respectively of (QP). ....(3p)

(c) For $x(\mu)$ and $\lambda(\mu)$ which you obtained in (3a), how does $\|x(\mu) - x^*\|_2$ and $\|\lambda(\mu) - \lambda^*\|_2$ behave when μ is small and positive? Is this as expected? Comment on the result. .... (2p)

4. Consider the nonlinear program

$$(NLP)\quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0,\ i = 1, 2, \quad x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2$, are twice-continuously differentiable. Assume specifically that $x^{(0)} = (0\ 0)^T$, at which it holds that

$$f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g_1(x^{(0)}) = 2, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

$$g_2(x^{(0)}) = -1, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(a) Your friend AF claims that since $\nabla f(x^{(0)}) = 0$ and $\nabla^2 f(x^{(0)}) \succ 0$, it must hold that $x^{(0)}$ is a local minimizer to (NLP). Explain why he is wrong. ...... (2p)

(b) We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)}$ be given above, let $\lambda^{(0)} = (0\ 0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. .... (8p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

5. Consider the semidefinite programming problem (P) defined as

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & G(x) \succeq 0, \end{array}$$

where $G(x) = \sum_{j=1}^n A_j x_j - B$ for B and A_j, $j = 1, \dots, n$, are symmetric $m \times m$-matrices. The corresponding dual problem is given by

$$(D) \qquad \begin{array}{ll} \text{maximize} & \operatorname{trace}(BY) \\ \text{subject to} & \operatorname{trace}(A_j Y) = c_j, \quad j = 1, \dots, n, \\ & Y = Y^T \succeq 0. \end{array}$$

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad c^T x - \mu \ln(\det(G(x))).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} c_j - \operatorname{trace}(A_j Y) &= 0, \quad j = 1, \dots, n, \\ G(x)Y - \mu I &= 0, \end{aligned}$$

assuming that $G(x) \succ 0$ and $Y \succ 0$ are kept implicitly. .... (5p)

(b) Show that a solution $x(\mu)$ and $Y(\mu)$ to the system of nonlinear equations, such that $G(x(\mu)) \succ 0$ and $Y(\mu) \succ 0$, is feasible to (P) and (D) respectively with duality gap $m\mu$. .... (3p)

(c) In linear programming, when $G(x)$ and Y are diagonal, it is not an issue how the equation $G(x)Y - \mu I = 0$ is written. The linearizations of $G(x)Y - \mu I = 0$ and $YG(x) - \mu I = 0$ are then identical. Explain why this is in general not the case for semidefinite programming and how it can be handled. .... (2p)

Remark: For a symmetric matrix M we above use $M \succ 0$ and $M \succeq 0$ to denote that M is positive definite and positive semidefinite respectively. You may use the relations

$$\frac{\partial \ln(\det(G(x)))}{\partial x_j} = \operatorname{trace}(A_j G(x)^{-1}) \quad \text{for} \quad j = 1, \dots, n,$$

without proof.

Good luck!