SF2822 Applied nonlinear optimization, final exam
Friday August 23 2013 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & 2e^{(x_1-1)} + (x_2 - x_1)^2 + 2x_3^2 \\ \text{subject to} & x_1x_2x_3 \le 2, \\ & x_1 + 2x_3 \ge c, \\ & x \ge 0, \end{array}$$

where c is a constant. Let $x^* = (1\ 1\ 1)^T$.

(a) Is there any value of c such that x^* satisfies the first-order necessary optimality conditions for (NLP)? ....(6p)

(b) Is there any value of c such that x^* is a global minimizer to (NLP)? ....(4p)

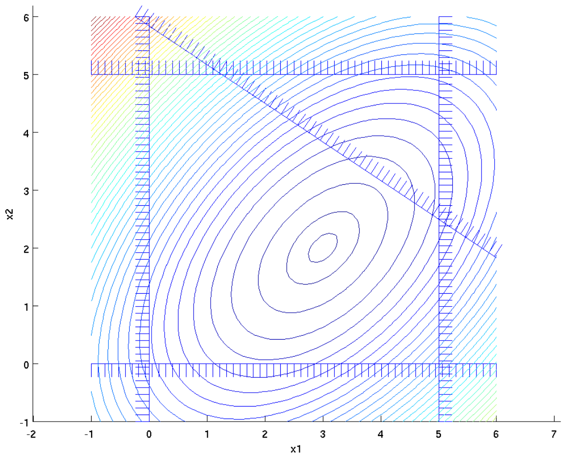
2. Consider the quadratic program (QP) defined by

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^THx + c^Tx \\ \text{subject to} & Ax \ge b, \end{array}$$

with

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -4 & -6 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -5 \\ -5 \\ -35 \end{pmatrix}.$$

The problem is illustrated geometrically in the figure below.



(a) Solve (QP) by an active-set method. Start at the point $x = (5 \ \ 0)^T$ with exactly one constraint in the working set, namely $-x_1 \geq -5$. You need not compute any numerical values, but you may utilize the fact that the problem is two-dimensional and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Exercise 2a, which is appended at the end. Motivate each step carefully. .... (5p)

(b) Solve (QP) with the same method as in Exercise 2a and with the same starting point, $x = (5 \ \ 0)^T$, but with $x_2 \geq 0$ as the only constraint in the working set instead. Illustrate your iterations in the figure corresponding to Exercise 2b, which is appended at the end. Motivate each step carefully. ....(5p)

3. Consider the nonlinear program

$$
(NLP) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_1(x) = 0, \\ & g_2(x) \geq 0, \\ & g_3(x) \geq 0, \\ & x \in I\!R^2, \end{array}
$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2, 3$, are twice-continuously differentiable. Assume specifically that we start at the point $x^{(0)} = (0 \ 0)^T$ with

$$
f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} -1 & -3 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
g_1(x^{(0)}) = 0, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} -1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
g_2(x^{(0)}) = 2, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},
$$

$$
g_3(x^{(0)}) = 4, \qquad \nabla g_3(x^{(0)}) = \begin{pmatrix} -1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_3(x^{(0)}) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}.
$$

In addition, assume that the initial estimate of the Lagrange multipliers, $\lambda^{(0)}$, are chosen as $\lambda^{(0)} = (0\ 1\ 0)^T$.

We assume that no linesearch is needed. The quadratic programming problems that arise may be solved in any way, that need not be systematic.

(a) Assume that we want to solve (NLP) by a sequential quadratic programming method, starting at $x^{(0)}$, $\lambda^{(0)}$. Perform one iteration, i.e. calculate $x^{(1)}$ and $\lambda^{(1)}$ given these values of $x^{(0)}$ and $\lambda^{(0)}$. .... (5p)

(b) Assume that the first constraint in (NLP) is replaced by an inequality, i.e., $g_1(x) = 0$ is replaced by $g_1(x) \geq 0$. Are $x^{(1)}$ and $\lambda^{(1)}$ the same? If not, calculate $x^{(1)}$ and $\lambda^{(1)}$ for this new problem. .... (3p)

(c) Assume that $\nabla^2 g_1(x) = 0$ for all x, i.e., the first constraint is linear. What, if anything, can be said about $g_1(x^{(1)})$ for each of the two different problems considered? .... (2p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

4. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

5. Consider the discrete optimization problem defined by

$$(DQP) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} & x_i \in \{0, 1\}, \quad i = 1, \dots, n, \end{array}$$

with $H = H^T \succeq 0$.

In theory, one approach for solving this problem would be to make a penalty transformation and approximately find a global minimizer of the problem

$$(DQP_\mu) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & \frac{1}{2} x^T H x + c^T x + \frac{1}{\mu} \sum_{i=1}^n x_i(1 - x_i) \\ \text{subject to} & 0 \leq x_i \leq 1, \quad i = 1, \dots, n, \end{array}$$

for a sequence of decreasing positive values of μ that tend to zero.

(a) Show that there is a value $\bar{\mu}$, with $\bar{\mu} > 0$, such that the objective function of (DPQ_μ) is nonconvex for $0 < \mu < \bar{\mu}$. .... (2p)

(b) Consider the one-dimensional problem, i.e.,

$$(DQP_\mu) \qquad \begin{array}{ll} \underset{x \in I\!R}{\text{minimize}} & \frac{1}{2} H x^2 + cx + \frac{1}{\mu} x(1 - x) \\ \text{subject to} & 0 \leq x \leq 1. \end{array}$$

Show that there is a value $\hat{\mu}$, with $\hat{\mu} > 0$, such that (DQP_μ) has one local minimizer at $x = 0$ and one local minimizer at $x = 1$ for $0 < \mu < \hat{\mu}$. Find the global minimizer of (DQP_μ) for such a value of μ. .... (3p)

(c) What do you think of the viability of solving the n-dimensional (DQP) by solving a sequence of problems of the form (DQP_μ)? Motivate your answer carefully. .... (3p)

(d) Show that (DQP) can be solved efficiently if H is diagonal. .... (2p)

Good luck!

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Figure for Exercise 2a:

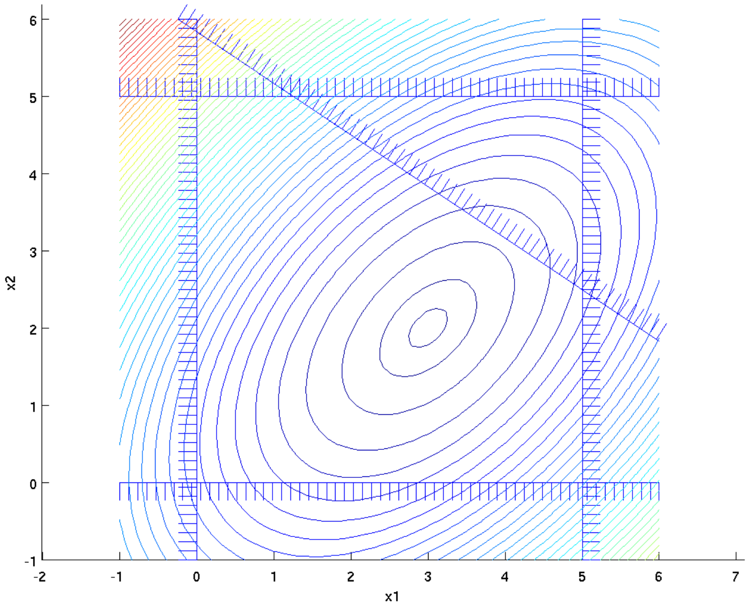

Figure for Exercise 2b: