SF2822 Applied nonlinear optimization, final exam
Wednesday June 3 2015 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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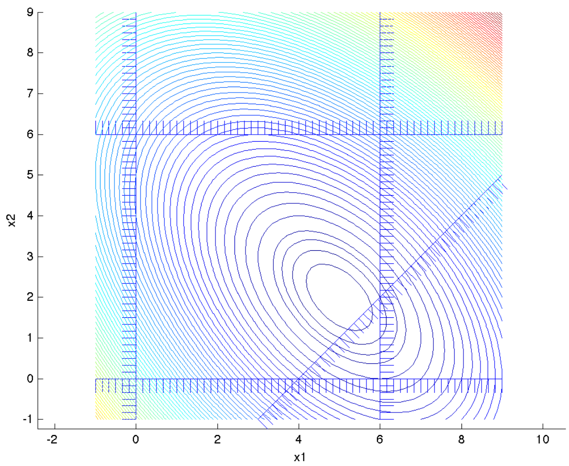
1. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -12 \\ -9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -4 \end{pmatrix}.$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (2 \;\; 0)^T$ with the constraint $x_2 \geq 0$ in the working set. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Question 1a which can be found at the last sheet of the exam. Motivate each step carefully. ....(4p)

(b) Assume that b_3 is changed from -6 to -4, so that the third constraint reads $-x_1 \geq -4$. Solve the corresponding quadratic program by an active-set method. Start at $x = (2\ \ 0)^T$ with no constraint in the working set. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Add the modified constraint and illustrate your iterations in the figure corresponding to Question 1b which can be found at the last sheet of the exam. Motivate each step carefully. ......(6p)

2. Consider the nonlinear program (NLP) given by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & (x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \\ \text{subject to} & \frac{3}{2} - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \geq 0. \end{array}$$

Assume that one wants to solve (NLP) by a sequential quadratic programming method for the initial point $x^{(0)} = (1\ 2)^T$ and $\lambda^{(0)} = 2$.

(a) Your friend AF claims that there is no need to perform any iterations. He claims that $x^{(0)}$ must be a global minimizer to (NLP), since (NLP) is a convex optimization problem and $\nabla f(x^{(0)}) = 0$. Explain why he is wrong. ...... (2p)

(b) Perform one iteration by sequential quadratic programming for solving (NLP) for the given $x^{(0)}$ and $\lambda^{(0)}$, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, and you do not need to perform any linesearch. ...... (8p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

3. Consider again the nonlinear program (NLP) of Question 2 given by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & (x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \\ \text{subject to} & \frac{3}{2} - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \geq 0. \end{array}$$

Assume that one wants to solve (NLP) by a primal-dual interior method.

Your friend AF insists on initializing by $x^{(0)} = (1\ 2)^T$ and $\lambda^{(0)} = 2$. (In accordance to the notation of the textbook, we define the Lagrangian as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.) He claims that this can be done, but unfortunately he has forgotten how.

Help AF set up a primal-dual interior method for which he can make use of the given $x^{(0)}$ and $\lambda^{(0)}$ as initial point. Let $\mu = 2$ and set up the linear system of equations to be solved at the first iteration for the given $x^{(0)}$, $\lambda^{(0)}$ and μ. First give the linear equations on general form and then insert numerical values. You need not solve the linear equations, but state how you would use the solution to generate the next iterate $x^{(1)}$, $\lambda^{(1)}$. ...... (10p)

Hint: You may want to introduce additional variables and/or constraints for which you need to set appropriate initial values.

4. Consider the semidefinite programming problem (P) defined as

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & G(x) \succeq 0, \end{array}$$

where $G(x) = \sum_{j=1}^{n} A_j x_j - B$ for B and A_j, $j = 1, \ldots, n$, are symmetric $m \times m$-matrices. The corresponding dual problem is given by

$$(D) \quad \begin{array}{ll} \text{maximize} & \operatorname{trace}(BY) \\ \text{subject to} & \operatorname{trace}(A_j Y) = c_j, \quad j = 1, \ldots, n, \\ & Y = Y^T \succeq 0. \end{array}$$

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad c^T x - \mu \ln(\det(G(x))).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} c_j - \operatorname{trace}(A_j Y) &= 0, \quad j = 1, \ldots, n, \\ G(x)Y - \mu I &= 0, \end{aligned}$$

assuming that $G(x) \succ 0$ and $Y \succ 0$ are kept implicitly. .... (5p)

(b) Show that a solution $x(\mu)$ and $Y(\mu)$ to the system of nonlinear equations, such that $G(x(\mu)) \succ 0$ and $Y(\mu) \succ 0$, is feasible to (P) and (D) respectively with duality gap $m\mu$. .... (3p)

(c) In linear programming, when $G(x)$ and Y are diagonal, it is not an issue how the equation $G(x)Y - \mu I = 0$ is written. The linearizations of $G(x)Y - \mu I = 0$ and $YG(x) - \mu I = 0$ are then identical. Explain why this is in general not the case for semidefinite programming and how it can be handled. .... (2p)

Remark: For a symmetric matrix M we above use $M \succ 0$ and $M \succeq 0$ to denote that M is positive definite and positive semidefinite respectively. You may use the relations

$$\frac{\partial \ln(\det(G(x)))}{\partial x_j} = \operatorname{trace}(A_j G(x)^{-1}) \quad \text{for} \quad j = 1, \ldots, n,$$

without proof.

5. Consider the indefinite quadratic program (QP) defined by

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2} x^T D x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \end{pmatrix},$$
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \end{pmatrix}.$$

You may throughout use the fact that D is diagonal.

(a) Determine how many negative eigenvalues D has. Also determine the maximum number of active constraints (QP) may have at any feasible point. Use these two facts to motivate why (QP) cannot have any local minimizer. ....(3p)

(b) Is it possible to add a bound-constraint to (QP) such that $(1\ 1\ 1\ 1)^T$ is a local minimizer to the resulting problem? If so, determine such a constraint....(7p)
Note: A bound-constraint is a constraint on the form $x_i \geq l_i$ or $-x_i \geq -u_i$, for some i, $i = 1, \ldots, 4$, where l_i or u_i is a given numeric value.

Good luck!

Name: .... Personal number: .... Sheet number: ....

Figure for Question 1a:

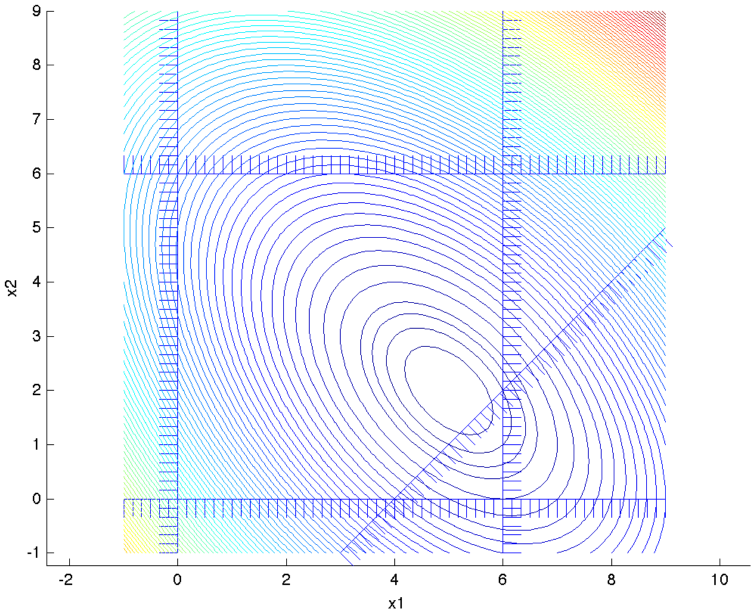


Figure for Question 1b: