**SF2822 Applied nonlinear optimization, final exam**
**Thursday August 20 2015 8.00–13.00**

*Examiner:* Anders Forsgren, tel. 08-790 71 27.

*Allowed tools:* Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

*Solution methods:* Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

*Note!* Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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**1.** Consider the nonlinear programming problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \ \geq 0, \\ & x \in I\!R^3, \end{array}$$

where $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R$ are twice continuously differentiable.

Assume that we have a point $x^*$ such that

$$f(x^*) = 5, \quad \nabla f(x^*) = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$g(x^*) = 0, \quad \nabla g(x^*) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T, \quad \nabla^2 g(x^*) = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Show that $x^*$ is not a local minimizer to $(NLP)$. ........................ (2p)

(b) Is it possible to add a bound-constraint to $(NLP)$ such that $x^*$ is a local minimizer to the resulting problem? If so, determine such a constraint....(8p)
*Note:* A bound-constraint is a constraint on the form $x_i \geq l_i$ or $-x_i \geq -u_i$, for some $i$, $i = 1, \ldots, 3$, where $l_i$ or $u_i$ is a given numeric value.

**2.** Consider the quadratic program $(QP)$ defined by

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & 2x_1 + x_2 \geq 1, \\ & x_1 + 2x_2 \geq 1, \\ & x_1 + x_2 \geq 2. \end{array}$$

Solve $(QP)$ with an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (1\ 2)^T$. The linear equations that arise may be solved in any way, that need not be systematic. ............................................................. (10p)

**3.** Consider the nonlinear program $(P)$ given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1-3)^2 + \frac{1}{2}(x_2-2)^2 \\ \text{subject to} & 2 - x_1^2 - \frac{1}{2}x_2^2 \geq 0. \end{array}$$

Assume that one wants to solve $(P)$ by a primal-dual interior method. A person named AF claims that a good initial point should be $x^{(0)} = (1\ 0)^T$ and let $\lambda^{(0)} = 2$. (In accordance to the notation of the textbook, we define the Lagrangian as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$.)

Let $\mu = 2$ and set up the linear system of equations to be solved at the first iteration for the given $x^{(0)}$, $\lambda^{(0)}$ and $\mu$. First give the linear equations on general form and then insert numerical values. You need not solve the linear equations, but state how you would use the solution to generate the next iterate $x^{(1)}$, $\lambda^{(1)}$. . . . . . . . . . . . . (10p)

**4.** Derive the expression for the symmetric rank-1 update, $C_k$, in a quasi-Newton update $B_{k+1} = B_k + C_k$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (10p)

**5.** Consider the nonlinear optimization problem $(NLP)$ defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1+1)^2 + \frac{1}{2}(x_2+2)^2 \\ \text{subject to} & 3(x_1+x_2-2)^2 + (x_1-x_2)^2 - 6 \geq 0. \end{array}$$

You have obtained a printout from an SQP solver for this problem, written by your not so reliable friend AF. The initial point is given by $x^{(0)} = (0\ 0)^T$ and $\lambda^{(0)} = 0$. Six iterations, without linesearch, have been performed. The printout reads:

| It | $x_1$ | $x_2$ | $\lambda$ | $\lVert\nabla f(x) - \nabla g(x)\lambda\rVert$ | $\lVert g(x)\rVert$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 2.2361 | 6 |
| 1 | 0.75 | $-0.25$ | $-0.14583$ | 0.74361 | 1.75 |
| 2 | 0.5285 | 0.050045 | $-0.20644$ | 0.098113 | 0.29052 |
| 3 | 0.57728 | 0.041731 | $-0.21804$ | 0.0044016 | 0.0081734 |
| 4 | 0.57666 | 0.043089 | $-0.21854$ | $4.1731 \cdot 10^{-6}$ | $5.5421 \cdot 10^{-6}$ |
| 5 | 0.57666 | 0.043089 | $-0.21854$ | $3.9569 \cdot 10^{-12}$ | $4.8512 \cdot 10^{-12}$ |
| 6 | 0.57666 | 0.043089 | $-0.21854$ | $1.1102 \cdot 10^{-15}$ | $1.7764 \cdot 10^{-15}$ |

(a) By looking at the values of $x^{(1)}$ and $\lambda^{(1)}$ computed by AF, explain to him why his solver cannot be working correctly. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

(b) Perform one iteration by sequential quadratic programming for solving $(NLP)$ for the given $x^{(0)}$ and $\lambda^{(0)}$, i.e., calculate correct values of $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, and you do not need to perform any linesearch. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (6p)

(c) What can you say about $x^{(1)}$, as calculated in Question 5b, with respect to $(NLP)$? . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

*Note:* According to the convention of the book we define the Lagrangian $\mathcal{L}(x,\lambda)$ as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function.

*Good luck!*