SF2822 Applied nonlinear optimization, final exam
Friday June 3 2016 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1+1)^2 + \frac{1}{2}(x_2+2)^2 \\ \text{subject to} & -2(x_1+x_2-1)^2 - (x_1-x_2)^2 + 10 = 0. \end{array}$$

You have obtained a printout from an SQP solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Eight iterations, without linesearch, have been performed. The printout reads:

It	x_1	x_2	λ	$\|\nabla f(x) - \nabla g(x)\lambda\|$	$\|g(x)\|$
0	0	0	0	2.2361	8.0000
1	-0.5000	-1.5000	0.1250	1.4577	9.0000
2	-0.2452	-1.0391	0.1061	0.1831	1.0668
3	-0.2319	-0.9490	0.1054	0.0195	0.0273
4	-0.2367	-0.9429	0.1045	0.0013	0.0001
5	-0.2371	-0.9426	0.1044	$6.5144 \cdot 10^{-5}$	$5.4934 \cdot 10^{-7}$
6	-0.2371	-0.9426	0.1044	$3.1220 \cdot 10^{-6}$	$1.3325 \cdot 10^{-9}$
7	-0.2371	-0.9426	0.1044	$1.4881 \cdot 10^{-7}$	$3.0447 \cdot 10^{-12}$
8	-0.2371	-0.9426	0.1044	$7.0819 \cdot 10^{-9}$	$8.8818 \cdot 10^{-15}$

(a) Formulate the first QP problem. Verify that the solution to this QP problem is given by the printout above. .... (6p)

(b) Show that the iterates converge to a global minimizer of (NLP). (You need not verify the numerical values.) .... (2p)

(c) The reason that you are asked to look at the printout is that the person who set up the solver is certain that the objective function and the first derivatives are evaluated correctly, but she is concerned that there might be an error in the evaluation of the second derivatives of the constraint function. Based on the printout, what do you think? .... (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x,\lambda)$ as $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function.

2. Consider the strictly convex bound-constrained quadratic program (QP) given by

$$(QP) \qquad \begin{array}{ll} \underset{x \in I\!R^3}{\text{minimize}} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & 0 \leq x_j \leq 1, \quad j = 1, 2, 3, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}.$$

Solve (QP) using an active-set method. Start at the point $x = (0\ 0\ 0)^T$ and let the constraints $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$ be in the working set. The linear equations that arise may be solved in any way that need not be systematic. Motivate each step carefully. .... (10p)

3. Consider the QP-problem (QP) defined as

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq -1. \end{array}$$

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytical expression for this small problem. .... (5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in (3a) converge to the optimal solution and Lagrange multiplier respectively of (QP). .... (3p)

(c) Make an approximation of $\lambda(\mu) - \lambda^*$ which is valid for small positive values of μ. Is this as expected? .... (2p)

4. Consider the NLP problem (P) defined as

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, \ldots, m, \\ & x \in I\!R^n, \end{array}$$

where f and g are twice-continuously differentiable.

A regular point with respect to the constraints is a point x^* such that $\nabla g_i(x^*)$, $i \in \{k : g_k(x^*) = 0\}$, are linearly independent.

(a) Formulate the second-order necessary optimality conditions for a regular point x^* to be a local minimizer for (P). .... (5p)

(b) For the special case when $g(x) = Ax - b$, prove the first-order necessary optimality conditions for a regular point x^* to be a local minimizer of (P). ..(5p)

5. Let $f(x,u)$ and $g_i(x,u)$, $i = 1,\dots,m$, be twice-continuously differentiable real-valued functions of $x \in I\!R^n$ and $u \in I\!R$. In particular, we are interested in the optimization problem

$$(NLP) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & f(x,\widetilde{u}) \\ \text{subject to} & g_i(x,\widetilde{u}) \ge 0, \quad i = 1,\dots,m, \end{array}$$

where x, as usual, is the vector of optimization variables and u is a parameter which is fixed to a particular value $\widetilde{u}$. (We will not be concerned with issues related to non-global minimizers, so you may assume that (NLP) is a convex problem.)

Let $\widetilde{x}$ be a regular point to (NLP) at which the second-order sufficient optimality conditions, with strict complementarity, hold for $\widetilde{x}$ together with a Lagrange multiplier vector $\tilde{\lambda}$, with $\tilde{\lambda} \in I\!R^m$.

The reason why u is introduced at all is that we have been asked to give an approximate expression for $z(u)$, the optimal objective function value as a function of u, for u near $\widetilde{u}$. Your not so reliable friend AF claims that

$$z(u) \approx z(\widetilde{u}) + \frac{\partial f(\widetilde{x},\widetilde{u})}{\partial u}(u - \widetilde{u}),$$

as this corresponds to the change in the objective function in the u-direction evaluated at $(\widetilde{x},\widetilde{u})$.

(a) Set up the first-order necessary optimality conditions for the problem

$$(NLP_u) \qquad \begin{array}{ll} \underset{x \in I\!R^n, u \in I\!R}{\text{minimize}} & f(x,u) \\ \text{subject to} & g(x,u) \ge 0, \\ & u = \widetilde{u}. \end{array}$$

.... (4p)

Remark: Note that (NLP) and (NLP_u) are equivalent, as although u is a variable in (NLP_u), it is fixed to $\widetilde{u}$.

(b) Use the result of Question 5a to determine if AF is right, alternatively give a correct approximate expression for $z(u)$, for u near $\widetilde{u}$. .... (6p)

Hint: Use sensitivity analysis.

Good luck!