SF2822 Applied nonlinear optimization, final exam
Thursday August 18 2016 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the quadratic program (QP) defined by

$$(QP) \quad \begin{array}{lll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 & \\ \text{subject to} & 2x_1 + x_2 + x_3 & \geq 3, \\ & x_1 + 2x_2 + x_3 & \geq 6, \\ & x_1 + x_2 + 2x_3 & \geq 3. \end{array}$$

Solve (QP) with an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (0\ 3\ 6)^T$. The linear equations that arise may be solved in any way, that need not be systematic. .... (10p)

2. Consider the same quadratic program (QP) as in Question 1, i.e.,

$$(QP) \quad \begin{array}{lll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 & \\ \text{subject to} & 2x_1 + x_2 + x_3 & \geq 3, \\ & x_1 + 2x_2 + x_3 & \geq 6, \\ & x_1 + x_2 + 2x_3 & \geq 3. \end{array}$$

Assume that we want to solve (QP) with a primal-dual interior point method. Also assume that we initially choose $x^{(0)} = (0\ 3\ 6)^T$, $\lambda^{(0)} = (1\ 2\ 3)^T$, and $\mu = 1$.

(a) When the constraints are in the form $Ax \geq b$, one may introduce slack variables s and rewrite the constraints as $Ax - s = b$, $s \geq 0$, when applying the interior method. Explain why this is not necessary for the given initial value $x^{(0)}$. (2p)

(b) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior point method for the given initial values. Formulate the general form and then introduce explicit numerical values into the system of equations. .... (5p)

(c) If the linear system of equations of Question 2b are solved, and the steps in the x-direction and the λ-direction are denoted by Δx and $\Delta\lambda$ respectively, one obtains

$$\Delta x \approx \begin{pmatrix} 1.9778 \\ -0.3556 \\ -2.9333 \end{pmatrix}, \quad \Delta\lambda \approx \begin{pmatrix} -0.9444 \\ -1.2778 \\ -1.8556 \end{pmatrix}, \quad A\Delta x \approx \begin{pmatrix} 0.6667 \\ -1.6667 \\ -4.2444 \end{pmatrix}.$$

Explain how you would use $x^{(0)}$, Δx, $\lambda^{(0)}$ and $\Delta\lambda$ to generate $x^{(1)}$ and $\lambda^{(1)}$. You need not give precise numerical values of $x^{(1)}$ and $\lambda^{(1)}$, but you should explain the principle and apply to this specific case. ....(3p)

3. Consider the problem (P) defined as

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) = 0, \\ & x \in I\!R^n, \end{array}$$

where $x \in I\!R^n$ and $g(x) \in I\!R^m$. The first-order necessary optimality conditions of (P) at a regular point take the form

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g(x) &= 0. \end{aligned}$$

Assume that Newton's method is used to solve this system of nonlinear equations in $x \in I\!R^n$ and $\lambda \in I\!R^m$. Derive the linear system of equations that needs to be solved for a given iterate (x_k, λ_k), at a Newton iteration. Show that this system of linear equations under suitable assumptions is equivalent to a certain QP problem. Give the assumptions and state the QP problem. ....(10p)

4. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & e^{x_1} - x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2 \\ \text{subject to} & -x_1^2 - x_2^2 - x_3^2 + 5 \geq 0, \\ & a^T x + 1 = 0, \end{array}$$

where $a \in I\!R^3$ is a given constant. Let $\widetilde{x} = (0\ 1\ 0)^T$.

(a) Determine a such that $\widetilde{x}$ fulfils the first-order necessary optimality conditions for (NLP). ....(6p)

(b) For the value on a which you determined in Question 4a, determine if $\widetilde{x}$ is a local minimizer to (NLP). ....(4p)

5. Consider the indefinite quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T D x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 2 \\ -1 \\ 2 \\ -1 \\ 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

You may throughout use the fact that D is diagonal.

(a) Determine how many negative eigenvalues D has. Also determine the maximum number of active constraints (QP) may have at any feasible point. Use these two facts to motivate why (QP) cannot have any local minimizer. ....(3p)

(b) Is it possible to add a bound-constraint to (QP) such that $(1\ 1\ 1\ 1\ 1)^T$ is a local minimizer to the resulting problem? If so, determine such a constraint. ....(7p)

Note: A bound-constraint is a constraint on the form $x_i \geq l_i$ or $-x_i \geq -u_i$, for some i, $i = 1, \ldots, 5$, where l_i or u_i is a given numeric value.

Good luck!