SF2822 Applied nonlinear optimization, final exam
Thursday June 1 2017 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the inequality-constrained quadratic program (IQP) defined by

$$(IQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$H = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \end{pmatrix}.$$

In this question, you may base your arguments on the fact that the problem has only one constraint. The linear systems of equations that may arise need not be solved in a systematic way.

(a) Consider the unconstrained quadratic program

$$(QP) \qquad \text{minimize} \quad \frac{1}{2}x^T H x + c^T x.$$

Is there a point that satisfies the second-order necessary optimality conditions for (QP)? .... (3p)

(b) Consider the equality-constrained quadratic program

$$(EQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax = b. \end{array}$$

Is there a point that satisfies the second-order necessary optimality conditions for (EQP)? .... (3p)

(c) Does (IQP) have a local minimizer? .... (2p)

(d) Does (IQP) have a global minimizer? .... (2p)

2. Consider the nonlinear program

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \;\; i = 1, 2, 3, \\ & x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2, 3$, are twice-continuously differentiable. Assume specifically that $x^{(0)} = (0\ 0)^T$, at which it holds that

$$f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$g_1(x^{(0)}) = -1, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$g_2(x^{(0)}) = -2, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$g_3(x^{(0)}) = -2, \qquad \nabla g_3(x^{(0)}) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_3(x^{(0)}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)}$ be given as above, let $\lambda^{(0)} = (0\ 0\ 1/2)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. . . . .(10p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

3. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. . . . . (10p)

4. Consider the QP-problem (QP) defined as

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq a, \end{array}$$

where a is a given constant scalar. The scalar a may take on any value, that may be positive, zero or negative.

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytical expression for this small problem. . . . .(5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in (4a) converge to the optimal solution and Lagrange multiplier respectively of (QP) for any given constant value of a. . . . . (5p)

5. Consider the optimization problem (P) defined by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2}x^T H x \\ \text{subject to} & x_j \in \{0, 1\}, \quad j = 1, \ldots, n, \end{array}$$

where H is an indefinite symmetric matrix. Problems of this type arise within combinatorial optimization, and the interest is to find a global minimizer.

One may compute lower bounds on the optimal value of (P) by considering relaxed problems.

(a) One way to relax (P) is to replace the constraints $x_j \in \{0,1\}$, $j = 1,\dots,n$, with $0 \le x_j \le 1$, $j = 1,\dots,n$. This gives a relaxed problem without discrete variables, according to

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & 0 \le x_j \le 1, \quad j = 1,\dots,n, \end{array}$$

Explain way this relaxed problem is not very interesting in practise. .... (3p)

(b) An alternative way to create a relaxation to (P) is to introduce a symmetric matrix Y and formulate the semidefinite programming problem

$$(SDP) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2}\,\text{trace}(HY) \\ \text{subject to} & \begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & Y = Y^T, \\ & y_{jj} = x_j, \quad j = 1,\dots,n. \end{array}$$

Show that if the constraint $Y = xx^T$ is added to (SDP), one obtains a problem which is equivalent to (P). .... (7p)

Hint: The following two results, which may be used without proof, might be useful:

(i) If H is an $n \times n$-matrix and x is an n-vector, then $\text{trace}(Hxx^T) = x^T H x$.

(ii) If Y is a symmetric $n \times n$-matrix and x is an n-vector, then

$$\begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if and only if} \quad Y - xx^T \succeq 0.$$

Good luck!