Linear dynamical systems

Consider

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

We assume $A(t)$, $B(t)$, $C(t)$, $D(t)$ are continuous over $(-\infty, \infty)$.

We discuss first solution to

$$\dot{x} = A(t)x$$
$$x(t_0) = a \qquad \text{— Cauchy problem}$$
$$t_0 \in \mathbb{R}, \quad a \in \mathbb{R}^n$$

Recall properties of solution to

$$\dot{x} = f(x, t)$$
$$x(t_0) = a$$

we typically use $x(t, t_0)$ to denote a solution

i.e. $\dot{x}(t, t_0) = f(x(t, t_0), t)$

Existence of solution: Continuity

Uniqueness of solution: Lipschitz continuity in x, and continuity in t.

Maximal solution: we know the solution exists on time interval $(t_1, t_2) \in R$

$t_1 = -\infty$, $t_2 = \infty$: the best case

Now let's go back to

$$\begin{cases} \dot{x} = A(t) x \\ x(t_0) = a \qquad (x(t_0, t_0) = a) \end{cases} \quad (*)$$

$\Rightarrow$ $A(t)x$ is Lipschitz in x

$\Rightarrow$ the solution is unique !

And the solution exists over $(-\infty, \infty)$!

we need to show $\| x(t_1, t_0) \| < \infty \ \forall t_1 \in R$

$t_1 > t_0$

Since $\|x\|^2 = x^T x$

$$\Rightarrow \quad \frac{d\|x(t, t_0)\|^2}{dt} = \dot{x}^T x + x^T \dot{x}$$

$$= x^T A^T(t) x + x^T A(t) x \qquad A_{n \times n}$$

$$\leq 2\|A(t)\| \, \|x\|^2 \qquad t \in [t_0, t_1]$$

$$\leq 2\|A(t^*)\| \, \|x\|^2$$

$\dot{z} = \alpha z \qquad z \in R$

$$\frac{dz}{dt} = \alpha z \Rightarrow \frac{dz}{z} = \alpha \, dt$$

$$\int_{t_0}^{t} \frac{dz}{z} = \int_{t_0}^{t} \alpha \, ds$$

$$\ln \frac{z(t)}{z(t_0)} = \alpha (t - t_0) \Rightarrow z(t) = e^{\alpha(t - t_0)} z(t_0)$$

By Comparison lemma, we have

$$\|x(t,t_0)\|^2 \le e^{2\|A(t^*)\|(t-t_0)}\|a\|^2$$

$$\Rightarrow \quad \|x(t,t_0)\| \le e^{\|A(t^*)\|(t-t_0)}\|a\| \quad \forall t \in [t_0, t_1]$$

Theorem: all solutions of $\dot{x} = A(t)x$ consists of a n-dimensional linear space.

Proof: Let $\Phi_k(t,t_0)$ $k=1,\cdots,n$ be solutions to $\dot{x} = A(t)x$ such that

$$\Phi_k(t_0,t_0) = e_k$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ kth position

$$[e_1 \cdots e_n] = I$$

Claim: $\Phi_1(t,t_0), \cdots, \Phi_n(t,t_0)$ are linearly independent $\forall t \in (-\infty, \infty)$

denote $\Phi(t,t_0) = [\Phi_1(t,t_0), \cdots, \Phi_n(t,t_0)]$

$$\Rightarrow \quad \Phi(t_0,t_0) = I$$

If $\Phi(t_1,t_0)$ is singular at some t_1

then the uniqueness of solution would be violated!

$\Rightarrow \quad \Phi(t, t_0)$ nonsingular $\forall t$

and $\Phi(t_0, t_0) = I$

This matrix is called state transition matrix for $\dot{x} = A(t)x$.

Example:

$$\dot{x}(t) = \begin{bmatrix} 0 & \cos(t) \\ 0 & 0 \end{bmatrix} x(t)$$

$$\Rightarrow \quad \dot{x}_1(t) = \cos(t)\, x_2(t)$$

$$\dot{x}_2(t) = 0$$

Let $\Phi_1(t_0) = \begin{bmatrix} x_1^1(t_0) \\ x_2^1(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\Phi_2(t_0) = \begin{bmatrix} x_1^2(t_0) \\ x_2^2(t_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Find $\Phi_1(t)$, $\Phi_2(t)$

with $\Phi_1(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2(t) = x_2(t_0) = 0 \quad \forall t$

$$\Rightarrow \quad x_1(t) = x_1(t_0) = 1 \qquad \forall t$$

$$\Rightarrow \quad \Phi_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

With $\Phi_2(t_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2(t) = x_2(t_0) = 1$, $\forall t$

$$\Rightarrow \quad \dot{x}_1 = \cos(t)$$

$$\Rightarrow \quad x_1(t) - \underbrace{x_1(t_0)}_{=0} = \int_{t_0}^{t} \cos(r)\,dr = \sin(t) - \sin(t_0)$$

$$\Rightarrow \quad \Phi_2(t) = \begin{bmatrix} \sin(t) - \sin(t_0) \\ 1 \end{bmatrix}$$

$\Rightarrow$ State transition matrix

$$\Phi(t, t_0) = \begin{bmatrix} 1 & \sin(t) - \sin(t_0) \\ 0 & 1 \end{bmatrix}.$$

We typically use t_0 to denote a given initial time, and use s to emphasize the involved time is "floating".

In this case we write $\Phi(t, s)$.

Now for the Cauchy problem

$$\begin{cases} \dot{x} = A(t)x \\ x(t_0) = a \end{cases}$$

Since $\quad a = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$$= a_1 e_1 + \cdots + a_n e_n$$

and $\quad a_1 \phi_1(t) + \cdots + a_n \phi_n(t)$ is also a solution to $\dot{x} = A(t)x$

and $\quad a_1 \phi_1(t) + \cdots + a_n \phi_n(t) = \Phi(t, t_0) a$

And $\Phi(t_0, t_0) a = a$

$\Rightarrow$ $x(t) = \Phi(t, t_0) a$ solves

$$\begin{cases} \dot{x} = A(t) x \\ x(t_0) = a \end{cases}$$

$\Rightarrow$ $\Phi(t, t_0) a$ is the solution.

Fundamental matrix:

Let $\psi_1(t), \cdots, \psi_n(t)$ be n linearly independent solutions to $\dot{x} = A(t) x$

$\Rightarrow$ We define $\underset{n \times n}{\Psi}(t) = [\psi_1(t), \cdots, \psi_n(t)]$

— fundamental matrix. (infinitely many)

$\Rightarrow$ $w_1 \psi_1(t) + \cdots + w_n \psi_n(t)$ is also a solution to $\dot{x} = A(t) x$

And $w_1 \psi_1(t) + \cdots + w_n \psi_n(t) = \Psi(t) w$

where $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

Let $a = \Psi(t_0) w$, $\Rightarrow w = \Psi^{-1}(t_0) a$

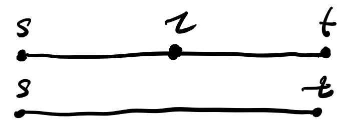
$\Rightarrow$ $\Psi(t) \Psi^{-1}(t_0) a$ solves

$$\begin{cases} \dot{x} = A(t) x \\ x(t_0) = a \end{cases}$$

$$\Rightarrow \boxed{\Phi(t,t_0)=\Psi(t)\Psi^{-1}(t_0)}$$

Properties of $\Phi(t,s)$: 0. $\dot{\Phi}(t,s)=A(t)\Phi(t,s)$

1. $\Phi(t,s)=\Phi(t,\tau)\Phi(\tau,s)$



Proof: Since $\Phi(t,s)=\Psi(t)\Psi^{-1}(s)$

$$\Rightarrow \Phi(t,s)=\Psi(t)\Psi^{-1}(\tau)\Psi(\tau)\Psi^{-1}(s)$$
$$=\Phi(t,\tau)\Phi(\tau,s)$$

2. $\Phi(t,s)$ is nonsingular and $\Phi^{-1}(t,s)=\Phi(s,t)$

Proof: Since $\Phi(t,s)=\Psi(t)\Psi^{-1}(s)$

$$\Rightarrow \Phi^{-1}(t,s)=(\Psi(t)\Psi^{-1}(s))^{-1}=\Psi(s)\Psi^{-1}(t)$$
$$=\Phi(s,t)$$

3. $\boxed{\dfrac{\partial\Phi(t,s)}{\partial s}=-\Phi(t,s)A(s)}$

Proof: Since $\Phi(t,s)=\Psi(t)\Psi^{-1}(s)$

$$\Rightarrow \Phi(t,s)\Psi(s)=\Psi(t)$$

$$\Rightarrow \frac{\partial\Phi}{\partial s}\Psi+\Phi\frac{\partial\Psi}{\partial s}=0$$

Since $\underline{\dot{\Psi}=A(t)\Psi}$

$$\Rightarrow \quad \frac{\partial \psi(s)}{\partial s} = A(s)\psi(s)$$

$$\Rightarrow \quad \frac{\partial \Phi}{\partial s}\psi(s) + \underline{\Phi A(s)}\,\psi(s) = 0$$

$$\Rightarrow \quad \frac{\partial \Phi}{\partial s} = -\Phi A(s)$$

Conclusion: any solution to $\dot{x} = A(t)x$

can be expressed as $\Phi(t, t_0)x(t_0)$

$$\left(\Phi(t, t_0)x(t_0)\right)^{\cdot} = \dot{\Phi}(t, t_0)x(t_0)$$

$$= A(t)\,\underline{\Phi(t, t_0)x(t_0)}$$

Now for

$$\dot{x} = A(t)x + B(t)u$$

Let $\quad z(t) = \Phi(t_0, t)x(t)$

$$\dot{z}(t) = \frac{\partial \Phi(t_0, t)}{\partial t}x(t) + \Phi(t_0, t)\dot{x}$$

$$= \underline{-\Phi(t_0, t)A(t)x(t)} + \underline{\Phi(t_0, t)(A(t)x} + B(t)u)$$

$$= \underline{\Phi(t_0, t)B(t)u(t)}$$

$$\Rightarrow \quad z(t_1) - z(t_0) = \int_{t_0}^{t_1} \Phi(t_0, s)B(s)u(s)\,ds$$

$$\Rightarrow \quad x(t) = \Phi^{-1}(t_0, t)z(t)$$

$$= \Phi(t, t_0)z(t)$$

$$= \Phi(t,t_0)x(t_0) + \Phi(t,t_0)\int_{t_0}^{t}\Phi(t_0,s)B(s)u(s)ds$$

$$= \boxed{\Phi(t,t_0)a + \int_{t_0}^{t}\Phi(t,s)B(s)u(s)ds}$$