Pole placement problem

Consider a reachable system as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & & 1 \\ -a_n & & \cdots & & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad , \quad x \in \mathbb{R}^n \quad (*)$$

$$T = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 0 & & \\ 1 & 0 \cdots & 0 \end{bmatrix}$$

Characteristic polynomial of A is

$$\chi_A(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$$

Let $u = kx = (k_1 \cdots k_n)x \Rightarrow$

$$A + bk = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & - & - & 1 \\ -a_n+k_1 & & \cdots & & -a_1+k_n \end{bmatrix}$$

$$\Rightarrow \quad \det(sI - (A+bk)) = s^n + (a_1 - k_n)s^{n-1} + \cdots + a_n - k_1$$

for any desired $\varphi(s) = s^n + \gamma_1 s^{n-1} + \cdots + \gamma_n$

$$\Rightarrow \quad k_n = a_1 - \gamma_1 , \; \cdots , \; k_1 = a_n - \gamma_n$$

thus, the pole placement problem is solved!

Now for any reachable system

$$\dot{x} = Ax + bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}$$

$$y = cx$$

Pole placement is solvable if $\exists \; \tilde{x} = Tx$

s.t. $\quad \tilde{A} = TAT^{-1} , \; \tilde{b} = Tb$

has the forms in $(*)$.

i.e. $$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \\ -a_n & \cdots & & -a_1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{c} = [1 \; 0 \cdots 0]$$

$$\Rightarrow \tilde{\Omega} = \begin{bmatrix} \tilde{c} \\ \vdots \\ \tilde{c}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} = I_{n\times n}$$

question: $\exists$ such a c and T ?

If c and T exist, $\Rightarrow \tilde{A} = TAT^{-1}$, $\tilde{b} = TB$, $c = \tilde{c}T$

$A = T^{-1}\tilde{A}T$

$$\Rightarrow \Omega = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} = \tilde{\Omega} T$$

$$\tilde{P} = [\tilde{b} \cdots \tilde{A}^{n-1}\tilde{b}] = TP$$

$$\Rightarrow \underbrace{T = \Omega = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}}$$

$$\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} [b \cdots A^{n-1}b] = \tilde{P} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ 0 & & & \vdots \\ 1 & \cdots & & 0 \end{bmatrix}$$

$$\Rightarrow c[b \cdots A^{n-1}b] = cP = [0 \; 0 \cdots 0 \; 1]$$

$$\Rightarrow c = [0 \cdots 0 \; 1] P^{-1}$$

$$\Rightarrow \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{i.e } Tb = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We assume $\sum_{i=1}^{n} \alpha_i C A^{i-1} = 0$ for some $\alpha_1, \cdots, \alpha_n$

i.e. $(\alpha_1 C + \alpha_2 CA + \cdots + \alpha_n CA^{n-1})b = 0$

$\Rightarrow \quad \alpha_n CA^{n-1}b = \alpha_n = 0 \quad \Rightarrow \alpha_n = 0.$

$\underbrace{(\alpha_1 C + \cdots + \alpha_{n-1} CA^{n-2})}_{Ab} = 0 \quad \Rightarrow \alpha_{n-1} = 0$

$\vdots$

$\alpha_1 = 0$

$\Rightarrow \quad T = \Omega$ is nonsingular!

$$\Rightarrow \quad \tilde{x} = Tx = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow \quad \tilde{x}_i = CA^{i-1}x, \quad i = 1, \cdots, n$$

$$\Rightarrow \quad \boxed{\begin{aligned} \dot{\tilde{x}}_i &= CA^{i-1}\dot{x}. \\ &= CA^{i}x + CA^{i-1}bu \end{aligned}}$$

$$\Rightarrow \quad \dot{\tilde{x}}_i = \tilde{x}_{i+1} \qquad i = 1, \cdots, n-1$$

$$\begin{aligned} \dot{\tilde{x}}_n &= CA^{n-1}\dot{x} \\ &= CA^{n}x + CA^{n-1}bu \\ &= CA^{n}x + u \end{aligned}$$

By Cayley-Hamilton theorem, $\chi_A(A) = 0$.

$$\Rightarrow \quad A^n + a_1 A^{n-1} + \cdots + a_n I = 0.$$

$$\Rightarrow \quad \dot{\tilde{x}}_n = C(-a_1 A^{n-1}x - \cdots - a_n x) + u$$

$$= -a_1 \tilde{x}_n - \cdots - a_n \tilde{x}_1 + u$$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ -a_n & & \cdots & & -a_1 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$\tilde{c} = [1 \; 0 \; \cdots \; 0]$$

Example: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $\quad \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{matrix}$

reachable.

I want $\varphi(s) = (s+r)(s+2r) = s^2 + 3rs + 2r^2$

$\Rightarrow \quad k_1 = -2r^2, \quad k_2 = -3r$

i.e $\quad u = -2r^2 x_1 - 3r x_2$

$\Rightarrow \quad A + bk = \begin{bmatrix} 0 & 1 \\ -2r^2 & -3r \end{bmatrix}$ $\qquad r > 0$

$s_1 = -r, \; s_2 = -2r$

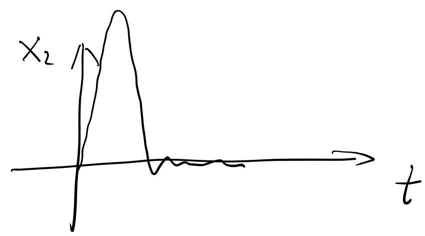
$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ -2r^2 & -3r \end{bmatrix} x$

$$e^{(A+bk)t} = \begin{bmatrix} 2e^{-rt} - e^{-2rt} & \frac{1}{r}(e^{-rt} - e^{-2rt}) \\ 2r(e^{-rt} - e^{-2rt}) & -e^{-rt} + 2e^{-2rt} \end{bmatrix}$$

if $\quad x_1(0) = 1, \; x_2(0) = 0.$

$x_2(t) = 2r(e^{-rt} - e^{-2rt})$

$x_2(t^*) = \frac{r}{2}$



We show now being reachable is sufficient for pole placement for multiple input system as well.

Consider

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

Assume (A, B) is reachable.

question: can we find $u = u_0 v$, $v \in \mathbb{R}$, $u_0 \in \mathbb{R}^m$

$$\Rightarrow \quad \dot{x} = Ax + (Bu_0)v$$
$$:= Ax + bv$$

s.t. (A, b) is also reachable. ?

Yes, we can do this if A is cyclic

Def: A is cyclic if each distinct eigenvalue of A has only one Jordan block.

Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Claim: if (A, B) is reachable, then

for all most all $u = Kx$, $A+BK$ is cyclic.

$\Rightarrow \quad u = Kx + u_0 v$

$\Rightarrow \quad \dot{x} = (A+BK)x + Bu_0 v$

$\Rightarrow \quad (A+BK, Bu_0)$ is reachable

then we can solve the problem by designing $v = kx$

$\Rightarrow \quad u = (K + u_0 k)x$

Now we show being reachable is also necessary for pole placement problem.

We assume $\exists\ u = Kx$ s.t. $A+BK$ has real and distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, s.t.

$\lambda_i > \|A\|$, $i = 1, \cdots, n$.

Let z_i be the eigenvector for λ_i, i.e.

$$(A+BK)z_i = \lambda_i z_i, \quad i = 1, \cdots, n$$

$[z_1 \cdots z_n]$ is nonsingular.

We show now $z_i \in \operatorname{Im} P$, $i = 1, \cdots, n$

$$\Rightarrow \quad (A - \lambda_i I)z_i = -BKz_i$$

$$\Rightarrow \quad (I - \frac{A}{\lambda_i}) z_i = BK \frac{z_i}{\lambda_i}$$

$$\Rightarrow \quad z_i = (I - \frac{A}{\lambda_i})^{-1} BK \frac{z_i}{\lambda_i}$$

$$\frac{1}{1-x}$$

$$= (I + \frac{A}{\lambda_i} + \frac{A^2}{\lambda_i^2} + \cdots) BK \frac{z_i}{\lambda_i}$$

$$= BK \frac{z_i}{\lambda_i} + ABK \frac{z_i}{\lambda_i^2} + \cdots$$

$$\Rightarrow \quad z_i \in \operatorname{Im}(B, AB, \cdots, A^{n-1}B, \cdots)$$

$$= \operatorname{Im}[B \cdots, A^{n-1}B] = \operatorname{Im} P, \quad i=1,\cdots,n.$$

$$\Rightarrow \quad \operatorname{Im} P = \mathbb{R}^n \quad \Rightarrow \text{reachable !}$$