KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2008
You may use min(3,(your score)/10) as bonus credit on the exam.

1. Determine a state feedback K such that the eigenvalues of the closed-loop system $\dot{x} = (A + BK)x$ are located in $\{-1, -2, -3\}$, for the case when

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

. . . . (5p)

Soultion: Since the system is controllable, it is standard to derive such a controller.

2. Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x,$$

where a is a constant.

(a) Can we always design a feedback controller $u = kx$ such that the closed-loop poles are placed in $\{-1, -1\}$? . . . . (3p)
Yes, since the system is controllable.

(b) Is the resulting closed-loop system observable? . . . . (3p)
Yes.

(c) Assume now that the state is not available. Can we always design an observer based controller that stabilizes the system, with the closed-loop poles located at $\{-1, -1\}$ and the observer dynamics having poles at $\{-2, -2\}$? . . . . . . (4p)
Not always, only when $a \neq 1$.

3. Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_1 - 2x_2 + u,$$

and the performance index

$$J = \int_0^\infty (2x_1^2 + x_2^2 + u^2)dt.$$

(a) Find all symmetric ARE solutions in terms of a. . . . . (4p)

(b) What is the optimal control $u = kx$? . . . . (3p)

Solution: Let $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$, $p_2^+ = -a + \sqrt{a^2+2}$, $p_2^- = -a - \sqrt{a^2+2}$. Then we have four solutions:

1. $p_2 = p_2^+, p_3 = -2 + \sqrt{5 + 2p_2^+}, p_1 = (p_2 + a)p_3 + 2p_2$.
2. $p_2 = p_2^+, p_3 = -2 - \sqrt{5 + 2p_2^+}, p_1 = (p_2 + a)p_3 + 2p_2$.
3. $p_2 = p_2^-, p_3 = -2 + \sqrt{5 + 2p_2^-}, p_1 = (p_2 + a)p_3 + 2p_2$.
4. $p_2 = p_2^-, p_3 = -2 - \sqrt{5 + 2p_2^-}, p_1 = (p_2 + a)p_3 + 2p_2$.

The first solution is positive definite, which should be used in the control. With this example we can see that solutions of ARE can be very complex even with 2nd order systems.

4. Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

(a) Assume P is a real positive **semidefinite** solution. Show that $kerP$ is A-invariant (i.e, $\forall x \in kerP,\ Ax \in kerP$) and $kerP \subset kerC$. . . . . (4p)

(b) Show that if (C, A) is observable, then every positive semidefinite solution P is positive definite. **Hint:** use the conclusions in (a) . . . . (4p)

Solution: Suppose $x \in KerP$. Multiplying both sides of the ARE by x:

$$PAx + C^T Cx = 0,$$

similarly $x^T C^T Cx = 0$. Therefore $x \in KerC$, which implies $kerP \subset kerC$. Furthermore, this leads to that $PAx = 0$. Thus $kerP$ is A-invariant.

When (C, A) is observable, the only A-invariant subspace in $KerC$ (unobservable subspace) is $\{0\}$. Thus, $kerP = \{0\}$.

5. Consider the discretized Newton's system with noise

$$\begin{aligned} x_1(t+1) &= x_1(t) + Tx_2(t) \\ x_2(t+1) &= v(t) \\ y(t) &= x_1(t) + w(t), \end{aligned}$$

where $T > 0$, $E\{v(t)\} = E\{w(t)\} = 0$, $E\{v(s)v(t)\} = E\{w(s)w(t)\} = 2\delta_{ts}$, and $E\{v(s)w(t) = 0$. Design Kalman filter for the system.

. . . . (5p)

Solution: This is a standard problem for Kalman filter.