KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2009
You may use min(3,(your score)/10) as bonus credit on the exam.

1. Solve the following linear state equations

(a) $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x(t),\ x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ....(2p) **Solution:** straight forward.

(b) $\dot{x}(t) = (1 - \cos(t))x(t),\ x(0) = 1.$ ....(1p)
Solution: straight forward.

(c) Let an $n \times n$ matrix $A(t)$ be continuous and $A(t) = P\Lambda(t)P^{-1}$, where P is constant and $\Lambda(t)$ is diagonal. Show the state transition matrix $\Phi(t,s) = exp(\int_s^t A(r)dr)$. ....(4p)
Solution: With such an A, $A(t)\int_{t_0}^t e^{A(r)}dr = A(t)\int_{t_0}^t Pe^{\Lambda(r)}P^{-1}$
$= P\Lambda(t)\int_{t_0}^t e^{\Lambda(r)}P^{-1} = P\int_{t_0}^t e^{\Lambda(r)}\Lambda(t)P^{-1} = \int_{t_0}^t e^{A(r)}dr A(t).$

2. Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & \lambda_1 & \lambda_3 \\ 0 & 1 & \lambda_2 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

where $\lambda_1, \lambda_2, \lambda_3$ are real numbers.

Solution: straight forward.

(a) Determine when the system is reachable ....(4p)

(b) Let $\lambda_2 = 0,\ \lambda_1 = \lambda_3 = 1$. Make a change of variables for which the system has the decomposed representation

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)$$

where (A_{11}, B_1) is reachable...(5p)

3. In this problem you will investigate whether it is possible to reconstruct the initial condition from measurements of the output for the system

$$\dot{x}(t) = Ax(t),\ x(0) = x_0$$
$$y(t) = Cx(t)$$

where

$$A = \begin{bmatrix} 0 & a \\ 1 & a \end{bmatrix},$$

where a is a constant.

(a) Suppose you can measure $x_1(t)$, namely $C = [1\ 0]$, for what a is it possible to reconstruct the initial state x_0? .... (2p)
Solution: $a \neq 0$.

(b) Suppose you can measure $x_2(t)$, namely $C = [0\ 1]$, for what a is it possible to reconstruct the initial state x_0? .... (2p)
Solution: any a.

(c) Suppose $a = 0$ and $C = [0\ 1]$. If you can measure $y(t)$ at $t = 0, \pi/2, \pi, 3\pi/2$, is it possible to reconstruct the initial state x_0? .... (3p)
Solution: Yes.

4. (a) Consider

$$\dot{x} = Ax,$$

where $x \in R^3$ and $A^T = -A$. For any $x(0) \neq 0$, show that the trajectory $x(t)$ lies on a sphere centered at the origin. .... (4p)
Solution: Since $\frac{d}{dt}\|x(t)\|^2 = x^T A^T x + x^T A x = 0$, $\|x(t)\|^2 = \|x(0)\|^2$.

(b) Consider the same system as in (a). We further require that for any $x_1(0) \neq 0$, the trajectory should rotate along (and only along) the x_3-axis as the time evolves. Give an example of such an A matrix. .... (3p)
Solution: For example, $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.