KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2009

1. Consider the pair (A, B), where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & \alpha \end{bmatrix}$$
$$B^T = \begin{bmatrix} 1 & -\alpha \end{bmatrix}.$$

For what α the Lyapunov equation $AP + PA^T + BB^T = 0$ has a positive definite solution? .... (4p)

Solution: $a \leq 0$ but not equal to -1.

2. You will in this problem derive and investigate a number of realizations for the transfer function

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+1} & \frac{1}{s+\alpha} \\ \frac{1}{s+1} & \frac{1}{s+\alpha} \end{bmatrix},$$

where α, γ are constant.

(a) Determine the standard reachable realization of $R(s)$. .... (3p)
Solution: One needs to discuss the cases $\alpha = 1$ and $\alpha \neq 1$.

(b) Discuss when the realization in (a) is observable? .... (3p)
Solution: When $\alpha = 1$ and $\gamma \neq 1$.

(c) Determine the standard observable realization of $R(s)$. .... (4p)

3. Suppose the following is a realization of a given $r(s)$:

$$(A, B, C) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

Solution: It is straight forward.

(a) Is this realization minimal? .... (3p)

(b) If not, use any method you know (for example, the Kalman decomposition) to find a minimal realization. .... (3p)

4. Consider

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx,\end{aligned}$$

where,

$$(A,B,C) = \left(\begin{bmatrix} 0 & 1 & 1 \\ -2 & -3 & \alpha \\ 0 & 0 & \beta \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \right),$$

and α, β are constant.

(*a*) When is A a stable matrix? .... (2p)
Solution: $\beta < 0$.

(*b*) When$A$ as defined above is not a stable matrix, is it still possible for the system to be BIBO stable? .... (3p)

Solution: Yes, when $\alpha = -2$.

5. A time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

is said to be internally balanced if A is a stable matrix and

$$\int_0^\infty e^{At}BB^Te^{A^Tt}dt = \int_0^\infty e^{A^Tt}C^TCe^{At}dt := W,$$

where W is a diagonal matrix with positive diagonal entries.

Consider the internally balanced system partitioned as

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u\\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.\end{aligned}$$

Show the reduced system

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u\\ \bar{y} &= \bar{C}\bar{x},\end{aligned}$$

where $\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\bar{B} = B_1 - A_{12}A_{22}^{-1}B_2$, $\bar{C} = C_1 - C_2A_{22}^{-1}A_{21}$, is also internally balanced. .... (8p)

Solution: The key in the proof is to express W as the solution to

$$AW + WA^T = -BB^T,$$

and

$$A^TW + WA = -C^TC.$$

Then from the fact that W is diagonal, one can show the conclusion.