KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2009
You may use min(3,(your score)/10) as bonus credit on the exam.

1. Determine a state feedback K such that the eigenvalues of the closed-loop system $\dot{x} = (A + BK)x$ are located in $\{-1, -2, -3, -4\}$, for the case when

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

.... (5p)

Solution: Since (A, B) is reachable, the calculation is straight forward.

2. Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

(a) Is the system observable? .... (1p)
Solution: Yes

(b) Design a feedback controller $u = kx$ such that the closed-loop poles are placed in $\{-1, -1\}$ .... (2p)
Solution: Straight forward.

(c) Is the resulting closed-loop system observable? Why? .... (2p)
Solution: No. State feedback can change observability.

(d) Assume now that the state is not available. Can we always design an observer based controller that stabilizes the system, with the closed-loop poles located at $\{-1, -1\}$ and the observer dynamics having poles at $\{-1, -1\}$? ...... (2p)
Solution: Yes, since the original system is both reachable and observable.

3. Consider

$$\dot{x}_1 = ax_1 + x_2$$
$$\dot{x}_2 = x_1 + ru,$$

where $a > 0,\ r > 0$. Given the cost function

$$J = \int_0^\infty (x_2^2 + u^2)dt,$$

let $u = -B^T Px$ denote the optimal control.

- What are the eigenvalues of $(A - BB^T P)$ as $r \to \infty$? .... (5p)
  Solution: In the ARE, let $\bar{p}_2 = rp_2,\ \bar{p}_3 = rp_3$. Then as $r \to \infty$, the ARE becomes

$$\begin{aligned} ap_1 - \frac{1}{2}\bar{p}_2^2 &= 0 \\ p_1 - \bar{p}_2\bar{p}_3 &= 0 \\ 1 - \bar{p}_3^2 &= 0. \end{aligned}$$

  After solving the equations we obtain the two closed-loop poles as $-\infty,\ -a$.

4. Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

- Assume P is a real positive **semidefinite** solution. Show that if (C, A) is observable, then $(A - BB^T P)$ is a stable matrix, i.e. all eigenvalues have negative real parts. .... (6p)
  Solution: We need first to show that P is positive definite if (C, A) is observable (see solution to last year's homework). The ARE can be rewritten as

$$(A - BB^T P)^T P + P(A - BB^T P) = -PBB^T P - C^T C.$$

  Then the same argument as we used in the lecture for proving Lemma 8.2.1 can be used here.

5. All conclusions about Kalman filter still hold if we replace $\mathcal{E}\{w(t)w^T(t)\} = R > 0$ by $\mathcal{E}\{w(t)w^T(t)\} = R(t) > 0$. Namely allow the covariance matrix for the noise to be time-varying.

Now consider the problem of measuring some constant scalar quantity x. Suppose initially nothing is known about x (i.e. $P(0) = \infty$). Then at each time instance $t = 0, 1, \cdots, n$, a measurement of x, $y(t)$, is made, with error covariance $r(t)$.

(a) Express the optimal estimation of x at t, $\hat{x}(t)$, which is based on measurements up to $t - 1$, by Kalman filter. .... (3p)
Solution: Straight forward.

(b) Show in the Kalman filter, $P(t + 1) < P(t)$. .... (2p)
Solution: Since $P(t + 1)^{-1} = P(t)^{-1} + r(t)^{-1}$.

(c) Write down the expression of $P(t)$ in terms of $r(i),\ i = 0, 1, \cdots, t - 1$. .... (2p)
Solution: $P(t)^{-1} = \sum_{i=0}^{t-1} r(i)^{-1}$.