KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Spring 2011
Disclaimer: For reference only.

1. Solve the following linear state equations

(a) $\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \lambda & -\sigma \\ 0 & \sigma & \lambda \end{bmatrix} x(t),\ x(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, where $\lambda \neq 0,\ \sigma > 0$. .... (2p)

Solution: omitted.

(b) $\dot{x}(t) = \frac{t}{1+t^2} x(t),\ \ x(t_0) = 1$. .... (2p)

Solution: omitted.

(c) Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

Show

$det\ \Phi(t, t_0) = e^{\int_{t_0}^{t} (a_{11}(s) + a_{22}(s)) ds}$. .... (3p)

Solution: Let $\Phi(t, t_0) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, then $det\ \Phi(t, t_0) = \phi_{11}\phi_{22} - \phi_{12}\phi_{21}$. $\frac{d}{dt}(det\ \Phi(t, t_0)) = (a_{11} + a_{22}) det\ \Phi(t, t_0)$, and $det\ \Phi(t_0, t_0) = 1$.

2. Determine when the system is reachable

(a)

$$\dot{x}(t) = \begin{bmatrix} \frac{t}{1+t^2} & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t),$$

where λ_1, λ_2 are real numbers. .... (3p)

Solution: The easiest way is to apply Lemma 3.1.3 in the compendium, which leads to the conclusion $\lambda_1 \neq \lambda_2$.

(b)

$$\dot{x} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)$$

where B_1 has *full row rank* and

$$A_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

and $a_1,\ a_2$ are real numbers. .... (3p)

Solution: It is equivalent to check when $[B_1^T\ 0]$ and $\begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix}$ is observable. Setting $y(t) = 0\ \forall t \geq 0$ implies $z_1(t) = 0$ since B_1^T has full column rank, which implies $A_3^T z_2(t) = 0$. $A_3^T z_2(t) = 0$ would implies $z_2(t) = 0$ if (A_3^T, A_4^T) is observable, or equivalently (A_4, A_3) is reachable, which requires $a_1 \neq a_2$.

3. Consider

$$\dot{x}(t) = Ax(t)$$
$$y(t) = cx(t)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}.$$

(*a*) Show that if the two eigenvalues of A are distinctive, then we can always find a c such that (c, A) is observable. .... (3p)
Solution: omitted.

(*b*) Show that if the imaginary part of the eigenvalues is none-zero, then (c, A) is observable for all $c \neq 0$. .... (2p)
Solution: We can show that A can be transformed into $\begin{bmatrix} \lambda & \sigma \\ -\sigma & \lambda \end{bmatrix}$, where $\sigma \neq 0$. $Det\ \Omega = (c_1^2 + c_2^2)\sigma$.

(*c*) Now let $a_{11} = a_{22} = 0$, $a_{12} = -a_{21} = \sigma > 0$ and suppose we can only measure $y(t)$ where $c \neq 0$ at $t = 0, T, 2T, ...$ What are the sampling periods T we should avoid if we want to reconstruct the initial state x_0 from the measurements? (2p)
Solution: We have a periodic system and it is obvious that the sampling period can not be divisible by the peoriod.

4. (*a*) A solution $x(t)$ is called periodic if $x(t + T) = x(t)\ \forall t$ for some $T > 0$. A periodic solution is called non-degenerated if $x(t)$ is not constant. Show that linear time-invariant systems

$$\dot{x} = Ax$$

can never have just one non-degenerated periodic solution. What further conclusion can you draw from your proof? .... (3p)
Solution: This can be shown by the fact that if $x(t)$ is a solution, then $kx(t)$ is also a solution for any scalar k. This fact also implies that a linear system can never have an isolated periodic solution (non-degenerated one of course).

(*b*) Verify that $X(t) = e^{At}X_0e^{Bt}$ is the solution to the matrix differential equation

$$\dot{X} = AX + XB,\ X(0) = X_0.$$

.... (2p)
Solution: omitted.