KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Spring 2011
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1. Consider the pair (C, A), where

$$A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(a) For what a_1 and a_2 the Lyapunov equation $A^T P + PA + C^T C = 0$ has a positive definite solution? . . . . (2p)
Solution: (C, A) should be observable, thus $a_2 \neq 0$; and A should be a stable matrix, thus $a_1 < 0$.

(b) Find the positive definite solution P. . . . . (3p)
Solution: Lyapunov equation leads to the following three equations:

$$a_1 p_{11} - a_2 p_{12} = 0, \;\; a_2 p_{11} + 2a_1 p_{12} - a_2 p_{22} = 0, \;\; a_2 p_{12} + a_1 p_{22} = -\frac{1}{2}.$$

We have $p_{11} + p_{22} = -\frac{1}{2a_1}$, $p_{12} = \frac{a_1}{a_2} p_{11}$, and $p_{11} = -\frac{a_2^2}{4a_1(a_1^2+a_2^2)}$. It is easy to verify that P is indeed positive definite.

2. This problem concerns stability as well.

(a) For $b = [1\ 1]^T$ and $c = [1\ 1]$, construct a 2×2 matrix A that is not asymptotically stable, but the resulting system is still BIBO stable . . . . (2p)

(b) Let $\dot{x} = Ax$, where
$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}.$$
When is the system stable, but not asymptotically stable? . . . .(1p)
Solution: Clearly at least one of the a_i has to be zero, otherwise the system is either asymptotically stable or unstable. Furthermore, the nonzero one has to be negative, otherwise the system will be unstable.

(c) Show that all the eigenvalues of A have real parts less than $-r < 0$ if and only if for a positive definite N

$$A^T P + PA = -N - 2rP$$

has a positive definite solution P. . . . . (2p)
Solution: All the eigenvalues of A have real parts less than $-r < 0$ iff $A + rI$ is a stable matrix, and the conclusion follows.

3. Consider

$$R(s) = \begin{bmatrix} \dfrac{k}{s+1} & \dfrac{1}{(s+1)^2} \\ \dfrac{2}{s+1} & \dfrac{1}{(s+1)^2} \end{bmatrix},$$

where k is a constant.

Solution: The keys to solving the problems are the following two polynomials, $\chi(s) = (s+1)^2$, and $\rho(s) = (s+1)^3$ if $k \neq 2$, otherwise $\rho(s) = (s+1)^2$.

(*a*) Determine the standard reachable realization of $R(s)$. .... (3p)

(*b*) Is the realization in (*a*) observable? .... (2p)

Solution: No. See also (d).

(*c*) Determine the standard observable realization of $R(s)$. .... (3p)

(*d*) What is the McMillan degree of $R(s)$? .... (2p)

Solution: $\delta(R) = 3$ if $k \neq 2$, otherwise $\delta(R) = 2$.

4. Suppose the following is a realization of a given $r(s)$:

$$(A, B, C) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 & 1 & 0 \end{bmatrix} \right)$$

(*a*) Show that the set of values of (c_1, c_2) such that the realization is not minimal is a line on the (c_1, c_2) plane. .... (4p)

Solution: The transfer function $r(s) = \frac{s^2+c_2s+c_1}{(s+1)^4}$. The realization is not minimal if there is zero/pole cancellation, or if $s^2 + c_2s + c_1 = s^2 + (1+a)s + a$, which implies $c_2 = c_1 + 1$.

(*b*) Is $(c_1, c_2) = (1, 2)$ on the line? If so, use Kalman decomposition to find a minimal realization. .... (6p)

5. Consider a time-invariant controllable system

$$\dot{x} = Ax + Bu,$$

where all the eigenvalues of A have positive real parts. Let $W = \int_0^\infty e^{-A\tau} BB^T e^{-A^T\tau} d\tau$. Show that for $u = -kB^TW^{-1}x,\ k > \frac{1}{2}$, the closed-loop system is asymptotically stable. .... (5p)

Solution: Since $(-A, B)$ is also controllable and $-A$ a stable matrix, $W = \int_0^\infty e^{-A\tau} BB^T e^{-A^T\tau} d\tau$ is positive definite and satisfies

$$-AW - WA^T + BB^T = 0,$$

i.e.

$$(A - kBB^TW^{-1})W + W(A - kBB^TW^{-1})^T + (2k-1)BB^T = 0.$$

When $2k-1 > 0$, $(A, \sqrt{2k-1}B)$ is controllable, thus $(A - kBB^TW^{-1}, \sqrt{2k-1}B) = (A - \sqrt{2k-1}B\frac{k}{\sqrt{2k-1}}B^TW^{-1}, \sqrt{2k-1}B)$ is controllable. By Corollary 4.3.6, $A - kBB^TW^{-1}$ is a stable matrix.