KTH Matematik

# Solution to Homework 3
## Mathematical Systems Theory, SF2832
## Spring 2011
## For reference only.

**1.** Determine a state feedback $K$ such that the eigenvalues of the closed-loop system $\dot{x} = (A + BK)x$ are located in $\{-1, -2, -3, -4\}$, for the case where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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**Solution:** Due to the structure of the system, one can design $u_1 = k_1(x_1, x_2)^T$ and $u_2 = k_2(x_3, x_4)^T$ separately for the following subsystems:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

**2.** Consider

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x.$$

(*a*) Is the system observable? . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (1p)

**Solution:** Yes.

(*b*) Design a feedback controller $u = kx$ such that the closed-loop poles are placed at $\{-1, -2, -3\}$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

**Solution:** omitted.

(*c*) Explain what happens to the observability of the closed-loop system. . . . . . (2p)

**Solution:** The closed-loop system is not observable since there is a pole/zero cancellation.

(*d*) Assume now that the full state is not available. Can we always design an observer-based control that stabilizes the overall system, with the closed-loop poles located at $\{-1, -2, -3\}$ and the observer dynamics having poles at $\{-1, -2, -3\}$? (1p)

**Solution:** Yes since the system is both controllable and observable.

**3.** Consider

$$\dot{x} = Ax + Bu$$

where $(A, B)$ is controllable and $A$ does not have any eigenvalue on the imaginary axis. Given the cost function

$$J = \int_0^\infty u^T u dt,$$

we can show that $u = -B^T Px$ is the optimal control, where $P$ is a positive *semi-definite* solution to the corresponding ARE.

(*a*) What are the eigenvalues of $(A - BB^T P)$ if $A$ is a stable matrix? ........ (1p)
**Solution:** Since $A$ is a stable matrix, $x(t)$ is bounded if $u = 0$. Thus $u = 0$ is a feasible control that apparently also minimizes the cost function. Therefore $(A - BB^T P) = A$.

(*b*) What are the eigenvalues of $(A - BB^T P)$ if $-A$ is a stable matrix? ...... (4p)
**Solution:** In order to make $A - BB^T P$ a stable matrix (thus $u = -B^T Px$ feasible), $P$ has to be positive definite by our discussion in Ch. 8. Since $A^T P + PA - PBB^T P = 0$ implies $A - BB^T P = -P^{-1}A^T P$, it has same eigenvalues as $-A^T$ thus as $-A$.

**Hint:** an optimal control is necessarily a feasible control.

**4.** Consider again

$$\dot{x} = Ax + Bu,$$

where $(A, B)$ is controllable. Given the cost function

$$J = \int_0^{t_1} u^T u dt + x(t_1)^T S x(t_1),$$

where $S$ is positive definite, and let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(*a*) Compute $\lim_{t_1 - t \to \infty} P(t_1 - t)$ for the case $A$ is a stable matrix. .......... (2p)
**Solution:** By using the adjoint system, we have $Y = exp(A^T(t_1 - t))S$, $X = exp(-A(t_1 - t)) + \int_0^{t_1 - t} exp(-As)BB^T exp(-A^T s)ds exp(A^T(t_1 - t))S$.
$P^{-1} = XY^{-1} = exp(-A(t_1 - t))S^{-1}exp(-A^T(t_1 - t)) + \int_0^{t_1 - t} exp(-As)BB^T exp(-A^T s)ds$.
If $A$ is stable, $P^{-1} \to \infty$; if $-A$ is stable, $P^{-1} \to \int_0^\infty exp(-As)BB^T exp(-A^T s)ds$, which satisfies $-P^{-1}A^T - AP^{-1} + BB^T = 0$.

(*b*) Compute $\lim_{t_1 - t \to \infty} P(t_1 - t)$ for the case $-A$ is a stable matrix. ........ (3p)
**Solution:** see (a).

(*c*) Comparing your results with those $P$ in Problem 3, what conclusion can you draw? ............................................................. (2p)
**Solution:** the limits are the same as those $P$ in Problem 3.

**Hint:** to determine $P$ is the same as determining $P^{-1}$ if $P$ is invertible.

**5.** Consider a second order random walk model

$$x(t+1) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} \frac{a^2}{2} \\ a \end{bmatrix} v(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + w(t),$$

where $v, w$ are uncorrelated white noises, with covariances $q,\ r$ respectively.

Design Kalman filter for the system. .......................................... (3p)

**Solution:** omitted.