SF2832 Mathematical Systems Theory
Solutions to Homework 1 (For reference only)

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1 State transition matrix

a)

$$\dot{x}(t) = Ax(t) = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x(t) \tag{1}$$

can be written as

$$\dot{x}_1 = x_2 \tag{2}$$
$$\dot{x}_2 = tx_2. \tag{3}$$

The solution for x_2 is

$$x_2(t) = x_2(t_0)e^{\frac{t^2-t_0^2}{2}} \tag{4}$$

and x_1 is

$$x_1(t) = x_2(t_0)\int e^{\frac{t^2-t_0^2}{2}}dt. \tag{5}$$

Now one can build the transition matrix from the solutions for the two initial states $x(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x(t_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$\Phi(t, t_0) = [\Phi_1(t, t_0), \Phi_2(t, t_0)] \tag{6}$$
$$= \begin{bmatrix} 0 & \int e^{\frac{t^2-t_0^2}{2}}dt \\ 0 & e^{\frac{t^2-t_0^2}{2}} \end{bmatrix} \tag{7}$$

b)

$$\dot{x}(t) = Ax(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2k & -2-3k & -3-k \end{bmatrix} x(t) \qquad \text{with } k > 2 \tag{8}$$

To determine the eigenvalues one can use the equation

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2k & -2-3k & -3-k-\lambda \end{vmatrix} = \lambda^2(\lambda + 3 + k) + 2k + \lambda(2 + 3k) = 0 \tag{9}$$

with

$$\lambda_1 = -2 \qquad \lambda_2 = -1 \qquad \lambda_3 = -k. \tag{10}$$

For $k > 2$ appears every eigenvalue once and the Jordan matrix J is a diagonal matrix. One can write A as

$$A = TJT^{-1} \tag{11}$$

with

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{12}$$

and

$$T = \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{k^2} \\ -\frac{1}{2} & -1 & -\frac{1}{k} \\ 1 & 1 & 1 \end{bmatrix}, \qquad T^{-1} = \frac{1}{(k-1)(k-2)} \begin{bmatrix} -4k(k-1) & 4-4k^2 & 4-4k \\ 2k(k-2) & k^2-4 & k-2 \\ 2k^2 & 3k^2 & k^2 \end{bmatrix}. \tag{13}$$

The state transition matrix for (8) is

$$\Phi(t) = e^{At} = e^{TJT^{-1}t} = Te^{Jt}T^{-1} = T\text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})T^{-1} \tag{14}$$

After this matrix multiplication one get the solution

$$\Phi(t) = \frac{1}{(k-1)(k-2)} \begin{bmatrix} -\frac{k(k-1)}{e^{2t}} + \frac{2k(k-2)}{e^t} + \frac{2}{e^{kt}} & -\frac{4k^2-4}{4e^{2t}} + \frac{k^2-4}{e^t} + \frac{3}{e^{kt}} & -\frac{4k-4}{4e^{2t}} + \frac{k-2}{e^t} + \frac{1}{e^{kt}} \\ \frac{2k(k-1)}{e^{2t}} - \frac{2k(k-2)}{e^t} - \frac{2k}{e^{kt}} & \frac{4k^2-4}{2e^{2t}} - \frac{k^2-4}{e^t} - \frac{3k}{e^{kt}} & \frac{4k-4}{2e^{2t}} - \frac{k-2}{e^t} - \frac{k}{e^{kt}} \\ -\frac{4k(k-1)}{e^{2t}} + \frac{2k(k-2)}{e^t} + \frac{2k^2}{e^{kt}} & -\frac{4k^2-4}{e^{2t}} + \frac{k^2-4}{e^t} + \frac{3k^2}{e^{kt}} & -\frac{4k-4}{e^{2t}} + \frac{k-2}{e^t} + \frac{k^2}{e^{kt}} \end{bmatrix}. \tag{15}$$

c)

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{16}$$

With $a_1^2 + a_2^2 + a_3^2 = 1$. Using the given hint one can show first

$$A^{2n+1} = (-1)^n A, \ n \geq 0 \tag{17}$$

and

$$A^{2n} = (-1)^{n+1} A^2, \ n \geq 1. \tag{18}$$

To show that (17) is satisfied one can bring A in the Jordan normal form. Its eigenvalues are:

$$\lambda_{1/2} = \pm\sqrt{-a_1^2 - a_2^2 - a_3^2} = \pm\sqrt{-1} = \pm i, \qquad \lambda_3 = 0 \tag{19}$$

and the Jordan normal form is

$$A = T\text{diag}(i, -i, 0)T^{-1} \tag{20}$$

and so

$$A^{2n+1} = (T\text{diag}(i, -i, 0)T^{-1})^{2n+1} = T\text{diag}(i^{2n+1}, (-i)^{2n+1}, 0)T^{-1}. \tag{21}$$

The diagonal elements i^{2n+1} and $(-i)^{2n+1}$ can be written as

$$i^{2n+1} = i(i^2)^n = i(-1)^n, \qquad (-i)^{2n+1} = -i(i^2)^n = -i(-1)^n \tag{22}$$

And so (21) shows that (17) is true:

$$A^{2n+1} = T\text{diag}(i(-1)^n, -i(-1)^n, 0)T = (-1)^n \underbrace{T\text{diag}(i, -i, 0)T^{-1}}_{A} \tag{23}$$

To show that (18) is true one can use an analog way as before:

$$A^{2n} = (T\text{diag}(i, -i, 0)T^{-1})^{2n} = T\text{diag}(i^{2n}, (-i)^{2n}, 0)T^{-1} \tag{24}$$

The diagonal elements i^{2n} and $(-i)^{2n}$ can be written as

$$i^{2n} = (i^2)^n = (-1)^{n-1} i^2, \qquad (-i)^{2n} = (i^2)^n = (-1)^{n-1}(-i)^2 \tag{25}$$

And so (24) shows that (18) is true:

$$A^{2n} = T\text{diag}((-1)^{n-1} i^2, (-1)^{n-1}(-i)^2)T = (-1)^{n-1} \underbrace{T\text{diag}(i^2, (-i)^2, 0)T^{-1}}_{A^2} \tag{26}$$

Now one can calculate the matrix exponential:

$$\begin{aligned}
e^{At} &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\frac{(At)^{2k+1}}{(2k+1)!} + \frac{(At)^{2k}}{(2k)!} \right) \\
&= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k A + I + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!}(-1)^{n+1} A^2 \\
&= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k A + I + \left(-\sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!}(-1)^n + 1 - 1 \right) A^2 \\
&= \underbrace{\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(-1)^k}_{\sin(t)} A + I + \Big(1 - \underbrace{\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}(-1)^n}_{\cos(t)} \Big) A^2 \\
&= I + A\sin(t) + A^2(1-\cos(t))
\end{aligned}$$

2 Inverted pendulum

a) The linearized equation is

$$\ddot{\theta} = \frac{g}{L}\theta - \frac{1}{L}\ddot{x} \tag{27}$$

with it's state space model for $x_1 = \theta$, $x_2 = \dot{\theta}$, $u = \ddot{x}$ and $y = \theta$:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix} u \tag{28}$$

$$y = Cx + Du = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u \tag{29}$$

b) The input-output description of the system is

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \tag{30}$$

$$= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} e^{A(t-s)} \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix} u(s)ds \tag{31}$$

with

$$e^{A(t-s)} = \frac{1}{2} \begin{bmatrix} e^{\sqrt{\frac{g}{L}}(t-s)} + e^{-\sqrt{\frac{g}{L}}(t-s)} & \sqrt{\frac{L}{g}} \left(e^{\sqrt{\frac{g}{L}}(t-s)} - e^{-\sqrt{\frac{g}{L}}(t-s)} \right) \\ \sqrt{\frac{g}{L}} \left(e^{\sqrt{\frac{g}{L}}(t-s)} - e^{-\sqrt{\frac{g}{L}}(t-s)} \right) & e^{\sqrt{\frac{g}{L}}(t-s)} + e^{-\sqrt{\frac{g}{L}}(t-s)} \end{bmatrix}. \tag{32}$$

After the the matrix multiplication we get solution for (31):

$$y(t) = \int_0^t -\sqrt{\frac{L}{g}} \frac{e^{\sqrt{\frac{g}{L}}(t-s)} - e^{-\sqrt{\frac{g}{L}}(t-s)}}{2} \frac{1}{L} u(s)ds \tag{33}$$

$$= -\frac{1}{\sqrt{Lg}} \int_0^t \sinh\left(\sqrt{\frac{g}{L}}(t-s) \right) u(s)ds \tag{34}$$

c) To show controllability for time invariant systems one can calculate the reachability matrix Γ and show that this has the rank n:

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} \tag{35}$$

$$= \left[\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix} \right] \tag{36}$$

$$= \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{L} & 0 \end{bmatrix} \tag{37}$$

$$\tag{38}$$

One can see that the matrix has full rank and so the system is controlable. To show observability we can use an analog way by using the observability matrix Ω:

$$\Gamma = \begin{bmatrix} C & CA \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \end{bmatrix} \tag{40}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

One can see that the row rank is 2 and so the system is also observable.

3 Control system for a spacecraft

The system of interrest is:

$$\dot{x} = Ax + Bu = \begin{pmatrix} 0 & \tilde{A} \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix} u \tag{42}$$

with

$$x = \begin{pmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix}, \tag{43}$$

a) For the time invariant system (42) one can show controllability like in section 2 with

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \tag{44}$$

$$\tag{45}$$

with

$$AB = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{46}$$

and

$$A^nB = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} \qquad \text{for } n > 1. \tag{47}$$

So the controllability matrix is

$$\Gamma = \begin{bmatrix} 0 & b & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

and if $b \in \mathbb{R}^{3x3}$ has rank 3, the controllability matrix has rank 6 and so the system is controlable. This is the case if the three vectors b_i are linear independent. An example for a possible combination of b is

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{49}$$

b) Using the result of the problem before one can set $b_3 = 0$. So b is a 3x2 matrix and can have a rank up to two. But the matrix b have to have rank 3 to make the system controllable. So the system with two pairs of gasjets is not controlable.