1a) For reference only. Courtesy of Mats Malmberg.

We will prove the necessity by contradiction.

Assume that rank $\begin{bmatrix} sI-A \\ C \end{bmatrix} < n$ for some s and that (C, A) is observable.

If rank $\begin{bmatrix} sI-A \\ C \end{bmatrix} < n$ for some s, then for this s $\exists a \in \mathbb{R}^n : \begin{bmatrix} sI-A \\ C \end{bmatrix} a = 0$

$$\Rightarrow \exists a \in \mathbb{R}^n : \begin{cases} Aa = Isa \\ Ca = 0 \end{cases}$$

$$\Rightarrow \Omega a = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} a = \begin{bmatrix} Ca \\ CsIa \\ \vdots \\ s^{n-1} Ca \end{bmatrix} = 0$$

Hence $a \in \ker \Omega$

But since Ω is assumed to be observable, $\ker \Omega = \{0\}$

We get a contradiction, thus rank $\begin{bmatrix} sI-A \\ C \end{bmatrix} = n$ $\forall s \in \mathbb{C}$ is a necessary condition for (C, A) to be observable.

3

1b)

Let A and C be defined by the given system.

Then
$$\begin{bmatrix} sI-A \\ C \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & s & -1 \\ a_1 & a_2 & \cdots & a_{n-1} & s+a_n \\ c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}$$

(C,A) is observable if and only if $\operatorname{rank}\begin{bmatrix} sI-A \\ C \end{bmatrix} = n, \ \forall s \in \mathbb{C}$

Let $x \in \mathbb{R}^n$ be any vector, then

$$\begin{bmatrix} sI-A \\ C \end{bmatrix} x = \begin{pmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & & \vdots \\ & & \ddots & \ddots & 0 \\ & & & s & -1 \\ a_1 & \cdots & & a_{n-1} & s+a_n \end{pmatrix} x = \begin{bmatrix} sx_1 - x_2 \\ sx_2 - x_3 \\ \vdots \\ sx_{n-1} - x \\ \sum_{i=1}^{n} a_i x_i + s x_n \\ \sum_{i=1}^{n} c_i x_i \end{bmatrix}$$

If $\exists x \in \mathbb{R}^n, x \neq 0 : \begin{bmatrix} sI-A \\ C \end{bmatrix} x = 0$, then $\operatorname{rank}\begin{bmatrix} sI-A \\ C \end{bmatrix} < n$ and (C,A) cannot be observable

suppose $\begin{bmatrix} sI-A \\ C \end{bmatrix} x = 0$, then by recursion on the first $n-1$ rows we get that $x_i = s^{i-1} x_1$. Using this we get the equations

$$\begin{cases} \sum_{i=1}^{n} a_i x_i + s x_n = 0 \\ \sum_{i=1}^{n} c_i x_i = 0 \end{cases} \Rightarrow \begin{matrix} (a_1 + a_2 s + \dots + a_n s^{n-1} + s^n) x_1 = 0 \quad (1) \\ (c_1 + c_2 s + \dots c_n s^{n-1}) x_1 = 0 \quad (2) \end{matrix}$$

We identify (1) as the characteristic polynomial, e.g. the roots are the eigenvalues of A.

Hence $\begin{bmatrix} sI-A \\ C \end{bmatrix} x = 0$ if and only if some root of $(c_1 + c_2 s + \dots + c_n s^n)$ is an eigenvalues of A, e.g. the kernel of $\begin{bmatrix} sI-A \\ C \end{bmatrix}$ is nonempty if and only if no root of the polynomial $c_1 + c_2 s + \dots + c_n s^n$ is an eigenvalue of A.

Thus by the property in 1a) (C,A) is observable if and only if no roots of $c_1 + c_2 s + \dots + c_n s^n$ is an eigenvalue of A

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1c)

From the solution of exersice 1b, in equation (1) and (2), we see that a feedback $(c, A+bk)$ will change these equations into

$$\begin{cases} \sum_{i=1}^{n} (a_i+k_i)x_i + sx_n = 0 \\ \sum_{i=1}^{n} c_i x_i = 0 \end{cases} \Rightarrow \begin{cases} \left((a_1+k_1 + (a_2+k_2)s + \dots + (a_n+k_n)s^{n-1} + s^n\right)x_1 = 0 \\ \left(c_1 + c_2 s + \dots + c_n s^{n-1}\right)x_1 = 0 \end{cases}$$

thus if we choose $c = [a, 0, \dots 0]$, $a \neq 0$ we can place the poles arbitrary and still maintain observability, according to the criterion in 1a.

2

2a) Show that the time-invariant system $\dot{x} = Ax$
$x \in \mathbb{R}^n$, $A^T = -A$ is not asymptotically stable

Solution:

The definition of asymptotical stability is given by definition 4.1.1

The system $\dot{x} = Ax$ is asymptotically stable if
$x(t) \to 0$ when $t \to \infty$, $\forall x_0$

Since the given system is time-invariant, we have that
$x(t) = \Phi(t, t_0)x_0 = e^{A(t-t_0)}x_0$

If $x(t)$ is asymptotically stable, then the norm
$\|x(t)\|^2 \to 0$ when $t \to \infty$, $\forall x_0$

$\|x(t)\|^2 = (x(t))^T x(t) = (e^{A(t-t_0)}x_0)^T(e^{A(t-t_0)}x_0) = e^{A^T(t-t_0)}e^{A(t-t_0)}x_0^2$

since $A^T = -A$ it is obvious that $A^T A = AA^T \iff (-A)A = A(-A)$
thus A^T and A commute, and we get that

$\|x(t)\|^2 = e^0 x_0^2 \neq 0$ as $t \to \infty$, $\forall x_0$

Hence the system is not asymptotically stable

2b)

Show that the system is (critically) stable

Solution:

The definition of being stable is given by definition 4.1.1

The system $\dot{x} = Ax$ is stable if the solution is bounded on the intervall $[0, \infty)$ for all initial values x_0

From 2a we have that

$\|x(t)\|^2 = x_0 \quad \forall t \in [0, \infty)$

thus the system is bounded $\forall x_0$ which, by definition proves stability.

2

3a)

Considering the system in 1b, with $n=2$ and pole placement in $s_1=-p$, $s_2=-2p$, by feedback control $u=kx+v$. We get the system.

$$\dot{x} = \tilde{A}x + Bv$$
$$y = Cx \quad , \quad \text{where } \tilde{A} = \begin{bmatrix} 0 & 1 \\ -3p & -2p^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [c_1, c_2]$$

If we use taylor expansion on x we get

$$x(t_0+h) = x(t_0) + \dot{x}(t_0)h + O(h^2)$$

by inserting $\dot{x}$ from our system we get

$$x(t_0+h) = x(t_0) + \tilde{A}x(t_0)h + Bvh + O(h^2)$$

Let $T_p = t_0+h$, and separate the matrix representation into two equations

$$\begin{cases} x_1(T_p) = x_1(t_0) + x_2(t_0)h + O(h^2) \\ x_2(T_p) = x_2(t_0) - 3p\,x_1(t_0)h - 2p^2x_2(t_0)h + v(t_0)h + O(h^2) = -3p\,x_1(t_0)h + x_2(t_0)[1-2p^2h] + vh + O(h^2) \end{cases}$$

Let $h = \frac{1}{p}$

$$\begin{cases} x_1(T_p) = x_1(t_0) + x_2(t_0)\cdot\frac{1}{p} + O(\frac{1}{p^2}) \\ x_2(T_p) = -3x_1(t_0) + x_2(t_0)[1-2p] + v(t_0)\frac{1}{p} + O(\frac{1}{p^2}) \end{cases}$$

let $p \to \infty$

then $x_1(T_p) \to x_1(t_0)$ and $x_2(T_p) \to -\infty$

Thus $\|x(T_p)\| \to \infty$ as $p \to \infty$

This shows that there are solutions $x(t)$ such that $|x(T_p)| \to \infty$ as $p \to \infty$ for some finite T_p

Intressant lösning!

2p

3b) given the system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ and the feedback $u = Ky + v$

We want to show that the reachable subspace $\mathcal{R}$ and the unobservable subspace $\ker \Omega$ are invariant under the feedback control.

By lemma 6.1.2 we have that the reachable subspace $\mathcal{R}$ is invariant under any feedback $u = \hat{K}x + v$

Thus set $\hat{K} = KC$ and the assumption is proven by lemma 6.1.2

To show that $\ker \Omega$ is invariant we note that by the fundamental theorem of linear algebra

$\mathbb{R}^n = \operatorname{Im}(\Omega^T) \oplus \ker \Omega$, where n is the dimension of A

and $(\operatorname{Im} \Omega^T)^{\perp} = \ker \Omega$

Thus if $\operatorname{Im} \Omega^T$ is invariant under the control, so is $\ker \Omega$.

We know that $\operatorname{Im} \Omega^T = \langle A^T / \operatorname{Im} C^T \rangle \triangleq \operatorname{Im}[C^T, A^T C^T, \ldots, A^{T(n-1)} C^T] =$

$$= \operatorname{Im} C^T + A^T \operatorname{Im} C^T + \ldots + A^{T(n-1)} \operatorname{Im} C^T$$

Let Ω_K be the observable subspace of the feedback controlled system

Then $\operatorname{Im} \Omega_K^T = \langle (A + BKC)^T | \operatorname{Im} C^T \rangle = \langle A^T + C^T K^T B^T | \operatorname{Im} C^T \rangle =$

$$= \langle A^T | \operatorname{Im} C^T \rangle + \langle C^T K^T B^T | \operatorname{Im} C^T \rangle$$

$\Rightarrow A^T \operatorname{Im} C^T \subseteq \operatorname{Im} \Omega^T$ (by lemma 3.2.7, setting $\operatorname{Im} \Omega^T = \mathcal{R}'$)

and $\operatorname{Im} C^T \subseteq \operatorname{Im} \Omega^T$

$\Rightarrow$ Introduce the notation $\tilde{A} = A^T$, $\tilde{B} = C^T$, $\tilde{K} = K^T B^T$, $\tilde{\mathcal{R}} = \operatorname{Im} \Omega^T$

Then again by Lemma 6.1.2, using $(\tilde{A}, \tilde{B}, \tilde{K})$ we have that
$\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_{\tilde{K}} \Leftrightarrow \operatorname{Im} \Omega^T = \operatorname{Im} \Omega_K^T$

$\Rightarrow \ker \Omega = \ker \Omega_K$ thus the unobservable subset is invariant under the given feedback control 3p

4 a) consider $R(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+1} & \frac{s+1}{s+2} \end{bmatrix}$

Determine the standard reachable realization of $R(s)$

Solution:

We start by breakig out the constant matrix D in $R(s)$

$$R(s) = \begin{bmatrix} \frac{s+1}{s+1} + \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+1} & \frac{s+1+1}{s+2} - \frac{1}{s+2} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+1} & \frac{-1}{s+2} \end{bmatrix}}_{R'(s) = C(sI-A)^{-1}B} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D}$$

For simplicity we will consider $R'(s)$ and then add our D when giving the final solution.

Next we thus want to find the least common denominator of $R'(s)$

$$\chi(s) = (s+1)(s+2) = s^2+3s+2 \Rightarrow r=2,\ a_1=3,\ a_2=2$$

Next we want to find matrices N_0, N_1.

$$\chi(s)R'(s) = \begin{bmatrix} s+2 & s+2 \\ 2s+4 & -s-1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}_{N_1} s$$

Thus we define our realization by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 4 & -1 & 2 & -1 \end{bmatrix}$$

4b)

By theorem 5.2.6 we have that:

The realization (A,B,C,D) of a matrix proper rational functions is reachable and observable if and only if it is minimal

The characteristic polynomial to $R(s)$ is equal to that of $R'(s)$. From 4a) we have that

$$R'(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+1} & \frac{-1}{s+2} \end{bmatrix}$$

The McMillan degree will give the dimension of a minimal realization. The McMillan degree is equal to the degree of the characteristic polynomial of $R(s) \Leftrightarrow$ degree of characteristic polynomial of $R'(s)$.

The only minor of order two has the least common denominator $(s+2)(s+1)^2$, which has a higher degree than all order one minors

Thus the McMillan degree $\delta(R(s)) = 3$, but the dimension of our realization in 4a is 4. Hence the realization is not minimal, and cannot be both reachable and observable. We know by construction that our realization is reachable, hence it is not observable

<u>answer:</u> No, the realization (A,B,C,D) from 4a is not observable.

4c)

As in 4a) we have that

$$R(s) = R'(s) + D, \quad R'(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+1} & \frac{-1}{s+2} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To construct an observable realization we start by a Laurentz expansion of $R'(s)$.

$$R'(s) = s^{-1} \begin{bmatrix} \frac{1}{1+s^{-1}} & \frac{1}{1+s^{-1}} \\ \frac{2}{1+s^{-1}} & \frac{-1}{1+2s^{-1}} \end{bmatrix} = s^{-1} \begin{bmatrix} \sum_{n=0}^{\infty} (-1)^n s^{-n} & \sum_{n=0}^{\infty} (-1)^n s^{-n} \\ 2\sum_{n=0}^{\infty} (-1)^n s^{-n} & -\sum_{n=0}^{\infty} (-2)^n s^{-n} \end{bmatrix}$$

$$\Rightarrow R'(s) = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}}_{R_1} s^{-1} + \underbrace{\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}}_{R_2} s^{-2} \ldots$$

We already know from 4a) that $\chi(s) = s^2 + 3s + 2 \Rightarrow a_1 = 3, a_2 = 2$

Thus our realization (A, B, C, D) is defined by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4d)

Refering to problem 4b) the McMillan degree

$\delta(R(s)) = 3$

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5 a)

Given the realization (A, B, C) we want to place the poles of the system at $\{-1,-1,-2,-2\}$, by feedback control $u = Kx$. We know that this is possible, since the given realization is controlabe.

Thus we get a new system $(\hat{A}, \hat{B}, \hat{C})$ where

$$\hat{A} = A + BK, \quad \hat{B} = B, \quad \hat{C} = C$$

To place the poles at $\{-1,-1,-2,-2\}$ is equivalent to

$$\varphi(s) = (s+1)^2(s+2)^2 = s^4 + 6s^3 + 13s^2 + 12s + 4$$

$$\Rightarrow \gamma_1 = 6, \ \gamma_2 = 13, \ \gamma_3 = 12, \ \gamma_4 = 4$$

We want $\hat{A}$ on the form

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma_4 & -\gamma_3 & -\gamma_2 & -\gamma_1 \end{bmatrix}$$

Let $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix}$, then we want to make $A + BK = \hat{A}$

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \\ k_5 & k_6 & k_7 & k_8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (k_1 + k_5) & (k_6 + k_2) & (1 + k_7 + k_2) & (1 + k_8 + k_4) \\ 0 & 0 & 0 & 1 \\ (k_5 - k_1) & (k_6 - k_2) & (1 + k_7 - k_3) & (1 + k_8 - k_4) \end{bmatrix}$$

$$\Rightarrow K = \begin{bmatrix} 2 & 6 & 7 & 3 \\ -2 & -6 & -7 & -4 \end{bmatrix}$$

$\therefore$ the feedback given by $u = \begin{bmatrix} 2 & 6 & 7 & 3 \\ -2 & -6 & -7 & -4 \end{bmatrix} x$ places the poles of the realization (A, B, C) in $\{-1,-1,-2,-2\}$

5b) We know that (A,B,C) is controllable.

If the system is also observable it is minimal.

$$\Omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$, which obviously does not have full rank

Hence the realization (A,B,C) is not minimal

To make the Kalman decomposition we want to find a base for $\ker \Omega$. By inspection $\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\} = \ker \Omega$

<u>Kalman decomposition:</u>

$$V_{\bar{o}r} = \mathcal{R} \cap \ker \Omega = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$$

$$\mathbb{R}^4 = V_{\bar{o}r} \oplus V_{or} \Rightarrow V_{or} = \text{span}\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$$\Rightarrow V_{\bar{o}\bar{r}} = V_{o\bar{r}} = \{0\}$$

Thus the matrices of the kalman decomposition are

$$A = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad B = \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{array}\right] \quad C = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \end{array}\right]$$

And a minimal realization is given by

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$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$