KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2013
For your reference only

1. Find the state transition matrix for the following systems

(a) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & t^2 \end{bmatrix} x(t)$

. . . . (2p)

Answer: omitted.

(b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -1-2k & -2-k \end{bmatrix} x(t),$

where the constant $k > 1$. . . . . (3p)

Answer: omitted.

(c) Let

$$\dot{x} = A(t)x$$

and

$$\dot{z} = K(t)z.$$

If $\Phi_z(t,s) = \Phi_x^T(s,t)$, where $\Phi(\cdot,\cdot)$ denote the state transition matrix, what is $K(t)$? . . . . (2p)

Answer: $K(t) = -A^T(t)$ since $\dot{\Phi}_x^T(s,t) = (\dot{\Phi}_x(s,t))^T = (-\Phi_x(s,t)A(t))^T$.

2. Consider the rotational motion of a point x in R^3 with respect to the origin:

$$\dot{x} = \omega(t) \times x,$$

where $\omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t))^T$ is the angular velocity, and "×" is the vector cross product.

(a) Express the kinematics of $x(t)$ in the form of $\dot{x} = A(t)x$. . . . . (2p)

Answer: $A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$.

(b) Use (a) to show that $\|x(t)\|^2 = \|x(t_0)\|^2 \ \forall t \geq 0$, i.e. the distance to the origin does not change over time (you are not supposed to use dot and cross products to show this). . . . . (3p)

Answer: $\|x(t)\|^2 = x(t_0)^T \Phi^T(t,t_0)\Phi(t,t_0)x(t_0)$. Since $A(t)^T = -A(t)$, we have $\Phi^T(s,t) = \Phi(t,s)$ (by 1.c above, $K(t) = -A(t)^T = A(t)$). Thus, $x(t_0)^T\Phi^T(t,t_0)\Phi(t,t_0)x(t_0) = x(t_0)^T\Phi(t_0,t)\Phi(t,t_0)x(t_0) = x(t_0)^T x(t_0)$.

3. Consider

$$\dot{x} = Ax + Bu, \ x \in R^n$$

where A and B are constant matrices. Show that if $x(0) \in \mathcal{R}$, then $x(t) \in \mathcal{R}$, $\forall t \geq 0$, and for all $u(t)$ such that the solution is unique. $\mathcal{R}$ is defined as $\mathcal{R} = Im\ (B, AB, \cdots, A^{n-1}B)$. (3p)

Answer: $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}x(0) + (B\int_0^t u(s) + \cdots + A^kB\int_0^t \frac{(t-s)^k}{k!}u(s)ds + \cdots$. $e^{At}x(0) \in \mathcal{R}$ if $x(0) \in \mathcal{R}$, and the second term is also in $\mathcal{R}$ by definition of $\mathcal{R}$ and Cayley-Hamilton Theorem.

4. Assume

$$\begin{aligned} \dot{x} &= Ax \\ y &= cx \end{aligned}$$

is observable, where $x \in R^n$ and $y \in R$.

(a) Let $\bar{x}_i = cA^{i-1}x, \ i = 1, \cdots, n$. What is $\bar{A}$ and $\bar{c}$ under the new coordinates? Use the characteristic polynomial of A to express elements of $\bar{A}$ if necessary. (3p)

Answer: Since the system is observable, the new coordinates are well defined. We can easily see that $\dot{\bar{x}}_i = cA^{i-1}\dot{x} = cA^i x = \bar{x}_{i+1}, \ i = 1, \cdots, n-1$, and $\dot{\bar{x}}_n = cA^n x = c(-a_nA^{n-1} - \cdots - a_1 I)x = -a_n\bar{x}_n - \cdots - a_1\bar{x}_1$. We use Cayley-Hamilton to derive the last equality.

(b) Show the n-tuple integrator system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u \\ y &= x_1 \end{aligned}$$

is both reachable and observable. .... (2p)

Answer: omitted.