KTH Matematik

**Homework 2**
**Mathematical Systems Theory, SF2832**
**Fall 2013**
**You may use min(5,(your score)/4) as bonus credit on the exam.**

**1.** (a) Consider a time-invariant system

$$\dot{x} = Ax,$$

where $x \in R^n$, $A$ is nilpotent ($A^k = 0$ for some $k$). For what nilpotent $A$ is $x = 0$ (critically) stable? . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (1p)

(b) Consider

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

1. For what $a_1$ and $a_2$ the Lyapunov equation $A^T P + PA + C^T C = 0$ has a positive definite solution? 2. Find the positive definite solution when it exists (2p)

**2.** Given the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{1}$$

$$y = cx = \begin{bmatrix} c_1 & c_2 \cdots & c_n \end{bmatrix} x, \tag{2}$$

(a) If $x(0) = 0$ and $u = \sin(t)$, discuss necessary and sufficient conditions on the coefficients of $c$ such that through observing $y(t)$ we can always draw the correct conclusion on if $A$ defined in (1) is a stable matrix. . . . . . . . . . . . . . . . . . . . . . . . (2p)

(b) Suppose $c_n = 1$ and $y(0) = 0$. Find a feedback control $u = Kx$ that forces $y(t) = 0$ $\forall t \geq 0$ in the closed-loop system. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (1p)

(c) Discuss necessary and sufficient conditions on $c_1, \cdots, c_{n-1}$ such that in the closed-loop system we obtain in 2.b, $y(t) = 0$, $\forall t \geq 0$ implies that $x(t) \to 0$ as $t \to \infty$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (2p)

**3.** Consider

$$R(s) = \begin{bmatrix} \frac{1}{s(s+2)} & \frac{1}{s+2} \\ \frac{k}{s(s+2)} & \frac{1}{s+2} \end{bmatrix},$$

where $k$ is a constant.

($a$) Determine the standard observable realization of $R(s)$. .................. (2p)

($b$) What is the McMillan degree of $R(s)$? .................................. (2p)

($b$) Let $k = 1$, derive a minimal realization. .................................. (2p)

**4.** This problem is about Lyapunov equations. Consider

$$\dot{x} = Ax + Bu.$$

Assume $(A, B)$ is controllable. Let $u = -B^T V^{-1}(0, t_1)x$, where $V(0, t_1) = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt$.

($a$) Show $\forall t_1 > 0$, $\bar{A} = A - BB^T V^{-1}(0, t_1)$ is a stable matrix. ............... (2p)

($b$) Show $\lim_{t_1 \to \infty} \bar{A} = A$ if $A$ self is a stable matrix. .......................... (2p)

($c$) What is $\lim_{t_1 \to \infty} \bar{A}$ if $-A$ is a stable matrix? ............................ (2p)